# The Euler-Lagrange Equation

### A Roller-coaster

The kinetic energy of the car $KE = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$

The potential energy of the car $PE = m\ g\ h(x)$

Newton's law of motion tells us that the acceleration of the car (in the x direction) is proportional to the effective horizontal force on the car and that this is equal to minus the potential gradient i.e.

$$a = m\ddot{x} = -\frac{d\,PE}{dx} = -m\ g\ \frac{dh(x)}{dx}$$

$$\ddot{x} = -g\ \frac{dh(x)}{dx} \tag{1}$$

If we multiply by -*m* integrate this equation with respect to *x* we get

$$m\ g\ h(x) = -m\int \ddot{x}\,dx = -m\int \frac{d\,\dot{x}}{dt}dx = -m\int \dot{x}\ d\,\dot{x} = -\frac{1}{2}m\,\dot{x}^2 + E$$

from which we deduce that:

$$E = KE + PE$$

Now it is a remarkable fact that there is a completely different way of deducing this result.

Instead of assuming Newton's Law, we assume the Euler-Lagrange equation which looks like this:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}} \tag{2}$$

where $L = KE - PE$

It is difficult to imagine a more different looking equation but it works. Here's how:

$$L = \frac{1}{2}m\,\dot{x}^2 - m\ g\ h(x)$$

$$\frac{\partial L}{\partial \dot{x}} = m\ \dot{x} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ \ddot{x}$$

$$\frac{\partial L}{\partial x} = -m\ g\ \frac{d\,h(x)}{dx}$$

so $m\,\ddot{x} = -m\ g\ \frac{dh(x)}{dx}$

as before.

I am struggling to understand *why* it works. Unlike $E$ which is the total energy, $L$ is the *difference* between the kinetic energy and the potential energy. When the roller-coaster is at the top of a hill, the KE is small and then PE is large; $L$ is therefore small (or even negative). At the bottom of the dips, $L$ is large. In general $L$ is a function of both the velocity of the car and its position along the ride (which determines its height) By using partial derivatives with respect to $\dot{x}$ and $x$, the Euler-Lagrange equation neatly separates out the kinetic and potential energies.

But this is no magic trick or coincidence. For deep reasons which I do not understand, the Euler-Lagrange equation is much more fundamental than Newton's laws. If you want to know the equations of motion for a particle moving under any system of conservative (i.e. energy conserving) forces, all you have to do is write down the Lagrangian expression, differentiate it a few times and there you are. What is more, it doesn't just work in Cartesian coordinates, it works in any coordinate system at all.

## Planetary orbits

Lets see if we can deduce the equations of motion for a planet orbiting the Sun. In this case, the potential function is an inverse power law so, in polar coordinates:

$$PE = -\frac{GMm}{r}$$

$$KE = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right)$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r} \tag{3}$$

Now since we have two independent coordinates, we have to evaluate two different E-L equations, namely:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

Taking the equation in r first:

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2} \tag{4}$$

which tells us that, in the absence of any angular rotation, the acceleration will be towards the Sun and proportional to $1/r^2$ as we should expect. The first term is, of course, the centrifugal acceleration. (It is correctly called centrifugal not centripetal because it is positive and directed away from the origin.)

If the motion is circular, then $\ddot{r} = 0$ and

$$\dot{\theta} = \sqrt{\frac{GM}{r^3}}$$

which gives us Kepler's third law: $T = 2\pi\sqrt{\frac{r^3}{GM}}$

Now taking the equation in $\theta$:

$$\frac{d}{dt}\left(m\, r^2\, \dot{\theta}\right) = 0$$

Integrating this w.r.t. gives us $m\, r^2\, \dot{\theta} = P$

where $P$ is a constant equal to the angular momentum of the system. (This is basically Kepler's second law).

We now have:

$$\dot{\theta} = \frac{P}{m\,r^2} \tag{5}$$

which tells us that as the planet swings closer to the Sun, its angular velocity speeds up.

Substituting equation (5) into equation (4) we get:

$$\ddot{r} = \frac{P^2}{m^2 r^3} - \frac{GM}{r^2} \tag{6}$$

which is basically the equation of motion of the orbit. If $r$ is large, the second term dominates and the planet accelerates back inwards; but as $r$ gets smaller the first term dominates and pushes it back out again.

Solving this equation is not a trivial exercise but the general solution is that of an ellipse:

$$r = \frac{a(1 - \epsilon^2)}{1 \pm \epsilon\cos\theta} \tag{7}$$

where $a$ is the semi-major axis, ε is the eccentricity and.

$$a(1 - \epsilon^2) = \frac{P^2}{GMm^2} \tag{8}$$

If the planet has its perihelion at $r = r_0$ ($r_0 < a$) then

$$r_0 = \frac{a(1 - \epsilon^2)}{1 + \epsilon} = a(1 - \epsilon) \tag{9}$$

hence

$$r_0(1 + \epsilon) = \frac{P^2}{GMm^2}$$

If the planet has a velocity $v_0$ at perihelion, then

$$P = m\,r_0\,v_0$$

$$r_0(1 + \epsilon) = \frac{r_0^2 v_0^2}{GM}$$

$$\epsilon = \frac{r_0 v_0^2}{GM} - 1$$

Now $\frac{r_0 v_0^2}{GM}$ is a dimensionless quantity which equals 1 when $v_0$ equals the circular orbit at that distance. We can therefore write:

$$\epsilon = \left(\frac{v_0}{v_c}\right)^2 - 1 \tag{10}$$

and

$$a = \frac{r_0}{1-\epsilon} = \frac{r_0}{2 - (v_0/v_c)^2} \tag{11}$$

It is also the case that:

$$T = 2\pi\sqrt{\frac{a^3}{GM}} \qquad (12)$$

A space station is orbiting the Earth in a circular orbit with a period of 92 minutes and radius 6500 km. It orbital speed is therefore approximately 7400 $ms^{-1}$. The crew jettison a canister of garbage from the rear of the station at a speed of 1 $ms^{-1}$. After one complete orbit the cannister will, presumably be approximately 5520 m behind the space station and after 7400 orbits, the station will catch up with the cannister again. Is this true?

If the speed is reduced by a small factor $p$ (= 1/7400 in this case), $a$ will be reduced by a factor of $2p$ because of the square in the denominator of equation (11). By similar reasoning $T$ will be reduced by a factor of $3p$ because $T$ is proportional to $a^{3/2}$. The cannister with therefore return after only 7400/3 = 2470 orbits or 158 days.

## *Parametric solution*

$$t = \frac{T}{2\pi}(\psi - \epsilon \sin\psi) \qquad (13)$$

where
$$T = 2\pi\sqrt{\frac{a^3}{GM}} \qquad (14)$$

$$x = a(\cos\psi - \epsilon) \qquad (15)$$

$$y = a\sqrt{(1 - \epsilon^2)}\sin\psi \qquad (16)$$

where $T$ is the period of the orbit, $a$ is the semi-major axis and ε is the eccentricity.

N.B. Ψ is NOT the angle at the centre of the ellipse.

We can eliminate ψ from equations (15) and (16) as follows

$$\cos\psi = \frac{x}{a} + \epsilon$$

$$\sin\psi = \frac{y}{a\sqrt{(1 - \epsilon^2)}}$$

$$\left(\frac{x}{a} + \epsilon\right)^2 + \frac{y^2}{a^2(1 - \epsilon^2)} = 1$$

$$(x + \epsilon a)^2 + \frac{y^2}{1 - \epsilon^2} = a^2 \qquad (17)$$

which is the equation for an ellipse whose focus is at the origin and whose centre is at the point (-ε$a$, 0)