The 15 Puzzle

Analysis

The 15 Puzzle looks like this:

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**14**	**15**	

Initially the tiles are all jumbled up (with the blank space in the bottom right hand corner) and by sliding the tiles around, the puzzle must be returned to the situation above. Since the action of sliding a block has the effect of moving the blank space one step up, down, left or right, the solution to the puzzle can be regarded as a 'walk' taken around the puzzle by the blank space back to its starting point. It is immediately obvious that this walk must involve an even number of moves.

We desire to show that of all the possible starting positions, exactly half are solvable and to derive a technique for determining whether a given starting position is solvable or not.

Let us start by considering an easier puzzle the 4 Puzzle:

B	**A**
C	

Note that I have labelled the cells cyclically starting from the blank cell in an anti-clockwise direction. The state I will call “ABC”. Let us consider the effect of walking the blank cell round the square in an anti-clockwise direction. After 1 circuit the state is “BCA”. After a second circuit the state is “CAB” and after the third circuit the puzzle reverts to its original state. Naturally, walking round in the other direction simply cycles through these three states in the opposite order.

Now it is easy to see that three possible states are omitted from this list, namely “ACB”, “BCA” and “BAC”. It is also significant that all three of the latter states can be reached by swapping just 2 of the letters whereas the former states can only be reached by swapping 2 pairs of letters.

Now consider a walk through the 15 Puzzle of the following type:

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**14**	**15**	

where the blank space walks through the following numbered cells – 15, 14, 10, 6, 7, 3, 2, 6, 10, 14, 15, and back to the start. It is clear that cells 15, 14, 10 an 6 will be returned to their starting point and that cells 7, 3 and 2 will be cycled. It is my conjecture – and strong belief – that every complex walk can be expanded to consist of a series of 'excursions' like the one above involving the cyclic alteration of three cells. This implies that that any complete walk can only be replaced by an **even** number of tile swaps.

This gives us a clue as to how to determine whether a given puzzle is solvable or not. Simply swap the 1 with the number in the top left hand corner (assuming that it is not in the right place). Repeat for all the other numbers until the puzzle is ordered again. If you have performed an even number of swaps, the puzzle is solvable; if an odd number, the puzzle is not solvable.

I am aware that the above argument is not mathematically watertight but I am confident that the result is correct.

Solving the puzzle

Can we use any of these insights to help with solving the puzzle? Well it is obviously quite unnecessary to break the solution down into a series of 'excursions' What you have to do is to is to 'line up' one or more numbers like a snake and then shunt them into place by moving the blank cell around a suitable loop. With each circuit the snake moves one space towards its goal. With a bit of ingenuity you can pick up more numbers along the way and shunt a whole line of numbers into place in one go. As the puzzle gradually gets sorted you will find yourself with less and less room in which to maneuver. One approach is to complete the top two lines first, then sort out cells 9 and 13.

When you get to cells 10 and 14 you are effectively solving a 3 × 2 puzzle. It often happens that the 10 and the 14 are in the right cells but the wrong way round. But you will find that it is always possible by doing a mixture of simple loops and 'figure of eight' loops to swap the 10 and the 14 and put them in the right place. A simple cycle will then put the 11 in the correct place and either the 12 and 15 will fall in the right places too or not. If the latter, then the original starting position was not solvable.