Random Walk in 1 dimension

The triangle of possibilities

At each step, the counter can move left or right with equal probability. After N steps the distribution of probabilities can be set out as follows:

N		-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8		*Divisor*
0										1										1
1									1	0	1									2
2								1	0	2	0	1								4
3							1	0	3	0	3	0	1							8
4						1	0	4	0	6	0	4	0	1						16
5					1	0	5	0	10	0	10	0	5	0	1					32
6				1	0	6	0	15	0	20	0	15	0	6	0	1				64
7			1	0	7	0	21	0	35	0	35	0	21	0	7	0	1			128
8		1	0	8	0	28	0	56	0	70	0	56	0	28	0	8	0	1		256

As with Pascal's triangle, the coefficients simply denote the number of different ways it is possible to reach that position in that number of steps by summing the two numbers in the row above to left and right.

The coefficients in the table can be calculated as follows:

$$c = \frac{N!}{\left(\frac{(N+S)}{2}\right)!\left(\frac{(N-S)}{2}\right)!}$$

For example, the coefficient at 7,3 is

$$C_7 = \frac{7!}{\left(\frac{(7+3)}{2}\right)!\left(\frac{(7-3)}{2}\right)!} = \frac{7!}{5!2!} = \frac{7.6}{2.1} = 21$$

and the probability of being at position P =3 after exactly 7 steps is 21/128.

Note that N+S must be even.

We can now calculate how likely it is that the walker will return to the origin in any given number of steps. For example, what is the probability the the walker will return to the start in 7 steps or fewer? It is obviously NOT equal to 2/4 + 6/16 + 20/64 because this equals 1.18 and probabilities can never be greater than 1. The mistake we have made is to include in the 20 possible ways of getting back to the origin in 6 steps those paths which have already passed through the origin in 4 or 2 steps. In effect we are counting these paths more than once. What we must do is exclude from the triangle all those additions which include contributions from the central column like this:

N		-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8		*Divisor*
0										1										1
1									1	0	1									2
2								1	0	(2)	0	1								4
3							1	0	1	0	1	0	1							8
4						1	0	2	0	(2)	0	2	0	1						16
5					1	0	3	0	2	0	2	0	3	0	1					32
6				1	0	4	0	5	0	(4)	0	5	0	4	0	1				64
7			1	0	5	0	9	0	5	0	5	0	9	0	5	0	1			128
8		1	0	6	0	14	0	14	0	(10)	0	14	0	14	0	6	0	1		256

So the probability of returning to the start in 7 or fewer steps is 2/4 + 2/16 + 4/64 = 0.6875

Suppose we wish to know the probability of reaching a certain position, say position 3, in 7 or fewer steps.

N		-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8		*Divisor*
0										1										1
1									1	0	1									2
2								1	0	2	0	1								4
3							1	0	3	0	3	0	(1)							8
4						1	0	4	0	6	0	3	0							16
5					1	0	5	0	10	0	9	0	(3)							32
6				1	0	6	0	15	0	19	0	9	0							64
7			1	0	7	0	21	0	34	0	28	0	(9)							128
8		1	0	8	0	28	0	55	0	62	0	28	0							256

This time we must exclude any additions stemming from the position 3 column.

The probability we desire is therefore 1/8 + 3/32 + 9/128 = 0.2891

Now although it is obvious that the probability of reaching a particular position in a finite number of steps is less than 1, it is not difficult to prove that you are certain to reach *every* position given an infinite number of steps. The proof is as follows:

Let us first ask the question: if we start at the origin, what is the probability of reaching position 1? Now the answer is not 0.5 because there are an infinite number of ways of reaching position 1 from the origin. We could go 0 > -1 > 0 > 1 or 0 > -1 > -2 > -1 > 0 > -1 > 0 > 1 etc. etc. and we must sum all of these probabilities together. Let us suppose that the answer is P_1.

Now the only way to get to position *P* is by passing first through position *P*-1 and the probability of reaching position *P* is therefore the probability of reaching position *P*-1 times the probability of getting from *P*-1 to *P*. But the probability of getting from *P*-1 to *P* is exactly the same as the probability of getting from the origin to position 1 i.e. P_1. It follows that the probability of getting from the origin to position *P* is equal to $(P_1)^P$.

Now for the cunning bit. If we start at the origin, we can get to position 1 in only two ways. Either we get there directly (and the probability of this is 0.5) or we first visit position -1 (probability 0.5) and then have to make two steps of progress back towards position 1 (probability $(P_1)^2$). What this means is that:

$$(P_1) = 0.5 + 0.5\,(P_1)^2$$

which admits of only one solution:

$$P_1 = 1$$

Random walks in higher dimensions

If we consider a random walk in two dimensions (think of a King on an infinite Chess board) the problem is a lot more complicated because to get to a certain position $P(x, y)$, there are four ways of approaching it, not just one, but, amazingly, the same result holds. Eventually the King will visit every single square on the board!

But when we move up to 3 and more dimensions the situation changes. There is so much space available that the chance of a random fly visiting a given finite region of space is less than 1 – despite having an infinite amount of time to fly around in! The probabilities have been calculated for spaces up to 8 dimensions as follows:

Dimensions of space	Probability of visiting any given cell after an infinite time
1	1.000
2	1.000
3	0.341
4	0.193
5	0.135
6	0.104
7	0.086
8	0.073

For large spaces the probability of recurrence becomes vanishingly small.