Pythagorean Triplets

Euclid's formula for generating all the primitive pythagorean triplets (PPT's)

$$a^2 + b^2 = c^2$$

Since a, b and c have no common factors, either a or b or both must be odd.

Suppose that a and b are both odd (and c is even). We can therefore write:

$$(2p + 1)^2 + (2q + 1)^2 = (2r)^2$$

Where p, q and r are integers.

In which case

$$4p^2 + 4p + 1 + 4q^2 + 4q + 1 = 4r^2$$
$$2(p^2 + p + q^2 + q) = 2r^2 - 1$$

Now the left hand side of this equation is even and the right hand side is odd. This is a contradiction so our assumption that a and b are both odd is wrong.

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Let us suppose that a is odd and b is even. This means that c must be odd. It also implies that $c + a$ and $c - a$ must both be even. We can therefore write

$$p = (c + a)/2$$
$$q = (c - a)/2$$

Where p and q are integers.

In which case

$$a = p - q$$
$$c = p + q$$

and

$$b = \sqrt{4pq}$$

Now if p and q have a common factor, a, b and c will share the same factor and the triplet will not be primitive. p and q must therefore be co-prime.

If that is the case, then in order for $4pq$ to have an integer square root, both p and q must be perfect squares. Suppose $p = m^2$ and $q = n^2$. Then

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

where m and n are co-prime, $m > n$ and one only is odd.

Here is a table of possible values of m and n.

m	n	a	b	c
2	1	3	4	5
3	2	5	12	13
4	1	15	8	17
4	3	7	24	25
5	2	21	20	29
5	4	9	40	41
6	1	35	12	37
6	3	27	36	45
6	5	11	60	61
7	2	45	28	53
7	4	33	56	65
7	6	13	84	85
8	1	63	16	65
8	3	55	48	73
8	5	39	80	89
8	7	15	112	113

(The entry for 6 3 has been greyed out because 6 and 3 are not co-prime.)

Another formula for generating all the primitive pythagorean triplets (PPT's)

Let us write $a = c - g$ and $b = c - h$ then

$$(c - g)^2 + (c - h)^2 = c^2$$
$$c^2 - 2cg + g^2 + c^2 - 2ch + h^2 = c^2$$
$$c^2 - 2(g + h)c + g^2 + h^2 = 0$$

from which we determine that

$$c = g + h + i$$
$$a = h + i$$
$$b = g + i$$

$$\text{where} \quad i = \sqrt{(2gh)}$$

[It is easy to show that a + b must be greater than c and therefore that the square root cannot be negative]

Suppose that g and h have a common factor p. Then i will also have the factor p. In fact, all three numbers a, b and c will have the same factor and the triplet will not be primitive. Hence g and h must be co-prime.

Since the expression must have an integral square root, then all the prime factors of g and h greater than 2 must appear in pairs and one of the two numbers must have an odd number of factors of 2.

These conditions can be met in the following way. Take any two co-prime integers u and v with u being odd. Then $g = u^2$, $h = 2v^2$ and $i = 2uv$

The following table gives the 16 possible values of u and v which generate triplets with $c < 100$.

u	v	g	h	i	a	b	c
1	1	1	2	2	3	4	5
1	2	1	8	4	5	12	13
1	3	1	18	6	7	24	25
1	4	1	32	8	9	40	41
1	5	1	50	10	11	60	61
1	6	1	72	12	13	84	85
3	1	9	2	6	15	8	17
3	2	9	8	12	21	20	29
3	4	9	32	24	33	56	65
3	5	9	50	30	39	80	89
5	1	25	2	10	35	12	37
5	2	25	8	20	45	28	53
5	3	25	18	30	55	48	73
5	4	25	32	40	65	72	97
7	1	49	2	14	63	16	65
7	2	49	8	28	77	36	85

Methods for generating particular series of triplets

Suppose we wish to generate all PPT's in which $c = a + 1$.

$$a^2 + b^2 = (a + 1)^2$$
$$b^2 = 2a + 1$$

This is easily achieved with

$$a = (b^2 - 1)/2$$
$$c = (b^2 + 1)/2$$

for all odd b.

Putting $b = 2p + 1$ we get:

$$a = 2n^2 + 2n$$
$$b = 2n + 1$$
$$c = 2n^2 + 2n + 1$$

n	a	b	c
1	4	3	5
2	12	5	13
3	24	7	25
4	40	9	41
5	60	11	61
6	84	13	85
7	112	15	113
8	144	17	145

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Suppose we wish to generate all PPT's in which $c = a + 2$.

$$a^2 + b^2 = (a + 2)^2$$
$$b^2 = 4a + 4$$

This is easily achieved with

$$a = (b^2 - 4)/4$$
$$c = (b^2 + 4)/4$$

Since this restrict b to even numbers, a (and c) must be odd. b must therefore be even. Writing $b = 2m$,

$$a = m^2 - 1$$
$$b = 2m$$
$$c = m^2 + 1$$

Since b is even, a must be odd which means that m must also be even. Writing $m = 2n$:

$$a = 4n^2 - 1$$
$$b = 4n$$
$$c = 4n^2 + 1$$

n	a	b	c
1	3	4	5
2	15	8	17
3	35	12	37
4	63	16	65
5	99	20	101
6	143	24	145
7	195	28	197
8	255	32	257

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Suppose we wish to generate all PPT's in which $c = a + k$. (k is odd and $k > 2$)

$$a^2 + b^2 = (a + k)^2$$
$$b^2 = 2ak + k^2$$

which gives

$$a = (b^2 - k^2)/2k$$
$$c = (b^2 + k^2)/2k$$

Now if k contains a certain prime factor p, b must also be divisible by p. It follows that b^2 will have two such prime factors and in consequence a and c will also have the same prime factor. If, on the other hand, k contains *two* prime factors and b one, they will cancel out. The argument can be extended to show that k can only contain an *even* number of individual prime factors - or, to put it another way, k must be a perfect square.

In which case b must be a multiple of $\sqrt{k}$

If we write $b = n\sqrt{k}$ then:

$$a = (n^2k - k^2)/2k = (n^2 - k)/2$$
$$c = (n^2k + k^2)/2k = (n^2 + k)/2$$

from which we see that n must be co-prime with k and n and k must have the same parity.

The following table shows some examples for $k = 9$ (with certain values of n which are not co-prime greyed out.)

k	n	a	b	c
9	5	8	15	17
	7	20	21	29
	9	36	27	45
	11	56	33	65
	13	80	39	89
	15	108	45	117
	17	140	51	149
	19	176	57	185

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Suppose we wish to generate all PPT's in which $c = a + k$ (k even and $k > 2$)

Write $k = 2j$ so that:

$$a^2 + b^2 = (a + 2j)^2$$
$$b^2 = 4aj + 4j^2$$

which gives

$$a = (b^2 - 4j^2)/4j = b^2/4j - j$$
$$c = (b^2 + 4j^2)/4j = b^2/4j + j$$

Now if j contains a certain prime factor p, b must also be divisible by p. It follows that b^2 will have two such prime factors and in consequence a and c will also have the same prime factor. If, on the other hand, j contains *two* prime factors and b one, they will cancel out. The argument can be extended to show that j can only contain an even number of individual prime factors - or, to put it another way, j must be a perfect square.

In which case b must be a multiple of $2\sqrt{j}$

If we write $b = 2n\sqrt{j}$ then:

$$a = 4n^2 j/4j - j = n^2 - j$$
$$c = 4n^2 j/4j + j = n^2 + j$$

from which we see that n must be co-prime with j.

In addition, since b is always even, n must not have the same parity as j. This is evident from the following table for $k = 18$

j	k	n	a	b	c
9	18	4	7	24	25
		6	27	36	45
		8	55	48	73
		10	91	60	109
		12	135	72	153
		14	187	84	205
		16	247	96	265
		18	315	108	333

The values of k which are valid are therefore:

$$1, 2, 8, 9, 18, 25, 49, 50, 72, 81, 98, 121, 128 \text{ etc.}$$

In the following table, Euclid's method has been used to generate a complete list of all the primitive pythagorean triplets together with their respective even and odd differences ($c - a$ and $c - b$ respectively).

m	n	a	b	c	k	k
2	1	3	4	5	2	1
3	2	5	12	13	8	1
4	1	15	8	17	2	9
4	3	7	24	25	18	1
5	2	21	20	29	8	9
5	4	9	40	41	32	1
6	1	35	12	37	2	25
6	5	11	60	61	50	1
7	2	45	28	53	8	25
7	4	33	56	65	32	9
7	6	13	84	85	72	1
8	1	63	16	65	2	49
8	3	55	48	73	18	25
8	5	39	80	89	50	9
8	7	15	112	113	98	1
9	2	77	36	85	8	49
9	4	65	72	97	32	25
9	8	17	144	145	128	1
10	1	99	20	101	2	81
10	3	91	60	109	18	49
10	7	51	140	149	98	9
10	9	19	180	181	162	1
11	2	117	44	125	8	81
11	4	105	88	137	32	49
11	6	85	132	157	72	25
11	8	57	176	185	128	9
11	10	21	220	221	200	1
12	1	143	24	145	2	121
12	5	119	120	169	50	49
12	7	95	168	193	98	25
12	11	23	264	265	242	1
13	2	165	52	173	8	121
13	4	153	104	185	32	81
13	6	133	156	205	72	49
13	8	105	208	233	128	25
13	10	69	260	269	200	9
13	12	25	312	313	288	1
14	1	195	28	197	2	169
14	3	187	84	205	18	121
14	5	171	140	221	50	81
14	9	115	252	277	162	25
14	11	75	308	317	242	9
14	13	27	364	365	338	1