Hopalong Fractals

Hopalong fractals are created by iterating a more or less simple formula in x and y over and over again. In general three types of behaviour can be distinguished. Often the series will diverge and the point will disappear off to infinity. Sometimes the point reaches a stable situation which repeats indefinitely with a finite period. Very occasionally, the point wanders around seemingly at random but remains within a finite set of points called a 'strange attractor'. Sometimes these attractors are extremely complex. This program allows you to choose between a number of different formulae each of which have up to 4 variables a, b, c and d which can be adjusted. Normally the starting point is (0,0) but by clicking on the picture you can initiate the run from anywhere. Points which are visited are either coloured white or in colour depending on the number of times that pixel has been visited. Facilities are provided which enable you to change the scale factor, display the origin and unit circle and vary the colours used.

In order to understand how these fractals are generated, we shall start with some simple generic cases which do not generate fractals.

Linear complex

$$x' = ax - by + c$$
$$y' = bx + ay$$

This is derived from the fundamental linear equation in complex numbers: Z' = AZ + C. What this equation essentially does is to take the point (x, y), rotate it by a certain angle (equal to the angle whose tangent is b/a), multiply it by a constant (equal to $\sqrt{(a^2 + b^2)}$) and add the constant c.

Using the default parameters (0.3, 0.95, 1), a 5 armed spiral converges on a stable point (0.5, 0.68) All points in the plane converge on this final point. In this case the angle whose tangent is 0.95/0.3 is about 72.5° - hence the five armed spiral.

If $b >= 1$ the formula becomes divergent for all $a > 0$. Try (0.1, 1, 0). Indeed, it is divergent whenever $a^2 + b^2 > 1$. If $a^2 + b^2) = 1$ then the series generates a circle but this cannot be called an *attractor* because a) it is unstable and b) different circles are generated depending on where you start from. (Just click on the main window to initiate the series from a new starting position.) The attractors we are looking for are stable and should result from a large selection of initial positions.

Quadratic complex

$$x' = x^2 - y^2 + a$$
$$y' = 2xy + b$$

(This is derived from the fundamental linear equation in complex numbers: $Z' = Z^2 + C$)

The default parameters put $a = -0.8$ and $b = 0$ (i.e. C = -0.8 + i0). If you click anywhere inside the corresponding Julia set (which is incidentally also a hopalong fractal generated by iterating the reverse function: $\sqrt{(Z - C)}$) you will generate a converging series. Outside the Julia set the series will diverge. (Technically, if you click exactly *on* the Julia set it will stay on the set but in practice it will fall off to one side or the other almost instantly.)

Now if you are familiar with the Mandelbrot set you will appreciate that the point (-0.8, 0) is inside lobe number 2 (the small lobe to the left of the main lobe on the axis.) The attractor here is a pair of points, not a single one. If you set *a* and *b* to (-0.5, 0) which is inside the main lobe of the Mandelbrot set, you will generate a single point attractor. Try setting *a* and *b* to (-0.15, 0.75) which is in lobe number 3 (the one at the top). How many points does the attractor consist of?

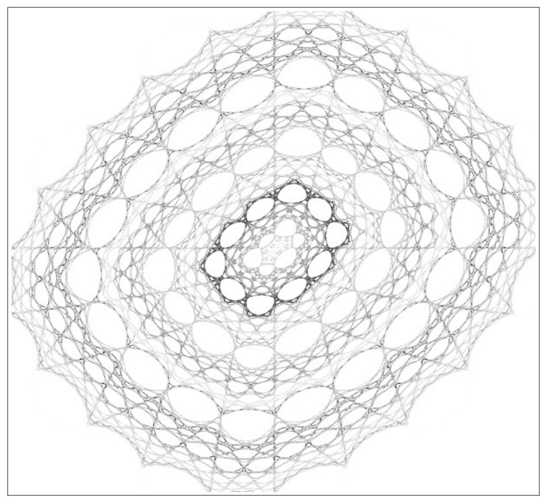
Simple ellipses

$$x' = y + f(x) \quad \text{where} \quad f(x) = (bx - c)$$
$$y' = -x + a$$

If you take a complex number ($x + iy$) and multiply it by $-i$ you get ($y - ix$) which turns x into y and y into $-x$. Multiplying by $-i$ rotates the point by 90° in an anti-clockwise direction. Repeat this a further 3 times and you get back where you started from. This is exactly what happens if you set *a*, *b* and *c* to zero - you always get 4 dots.

If you set *b* somewhere between 0 and 2, you will get an ellipse. What *b* seems to do is alter the angle of rotation which, provided that the angle is not a whole fraction of a circle, will generate a solid ellipse. (Set $b = 1$ for an exception to this rule.) Remarkably these ellipses are very stable and are not affected very much by changing *a* and *c*, only serving to change the size and position of the ellipse

The Classic Barry Martin fractal



$$x' = y + f(x) \quad \text{where} \quad f(x) = -\,\mathrm{SGN}(x)\sqrt{|bx - c|}$$
$$y' = a - x$$

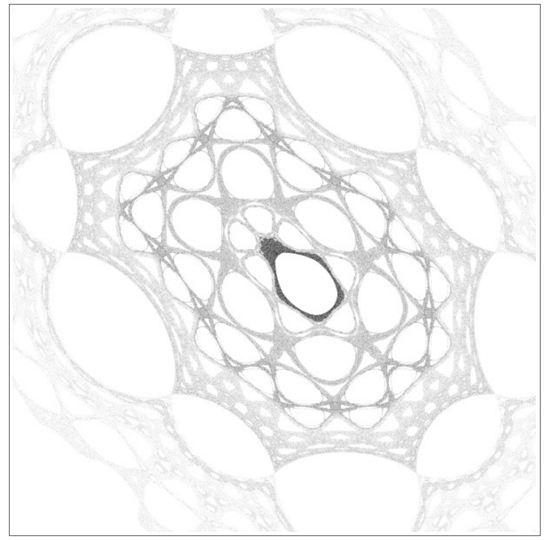
This is where the fun starts! This fractal discovered by Barry Martin in 1986 is essentially a modification of the simple ellipse. The only difference is the change to the function $f(x)$. Don't ask me why it works - or how Barry Martin came up with the idea. Just have fun!.

One thing to note about these patterns is that while they are certainly fractal they are not strictly *attractors* because you will find that the pattern generated depends critically upon the initial starting point. In this respect they resemble the simple ellipses produced by the previous algorithm. Each pattern consists of an infinite number of points and if you start on any of these points, you will generate the same pattern. But if you start from a point a tiny fraction away from the first one, you will generate something completely different. Sometimes you will get a complex fractal and sometimes a chain of loops. If you are lucky and hit the fractal you will find that eventually it turns back into the default fractal which is generated by starting at the origin.

One characteristic feature of Barry Martin fractals is that they seem to grow without limit. If this is true, they are not periodic (unlike the ellipses) but this is impossible to prove using a computer with a finite resolution because eventually the computer is going to return so close to a point already visited that its rounding error will force equality.

What all this seems to suggest is that any particular set of parameters divides the whole plane into two sets of points; one set (which includes the origin) is a fractal set which ranges over the whole plane and in which each point is only ever visited once and once only; and a second set of points which are periodic or quasi-periodic - that is to say their orbits are restricted to certain linear structures.

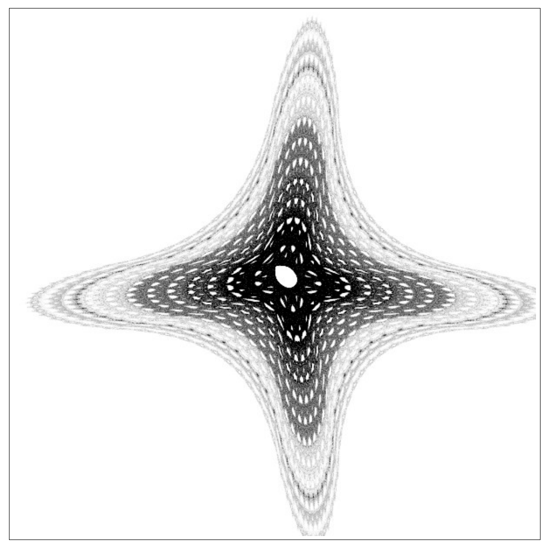
The positive Barry Martin fractal



$$x' = y + f(x) \quad \text{where} \quad f(x) = +\,\mathrm{SGN}(x)\sqrt{|bx - c|}$$
$$y' = a - x$$

Changing the sign of the function $f(x)$ generates similar but different fractals.

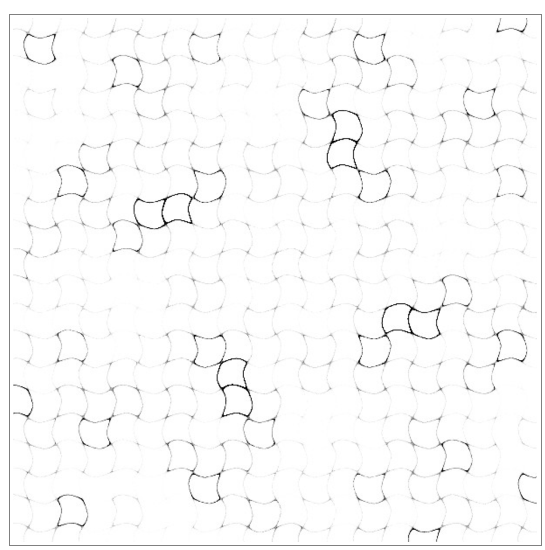
The additive Barry Martin fractal



$$x' = y + f(x) \quad \text{where} \quad f(x) = +\,\sqrt{|bx - c|}$$
$$y' = a - x$$

Here only the positive root is used. (Using the negative root produces the same fractals but reversed.)

The sinusoidal Barry Martin fractal



$$x' = y + f(x) \quad \text{where} \quad f(x) = \mathrm{SIN}(bx - c)$$
$$y' = a - x$$

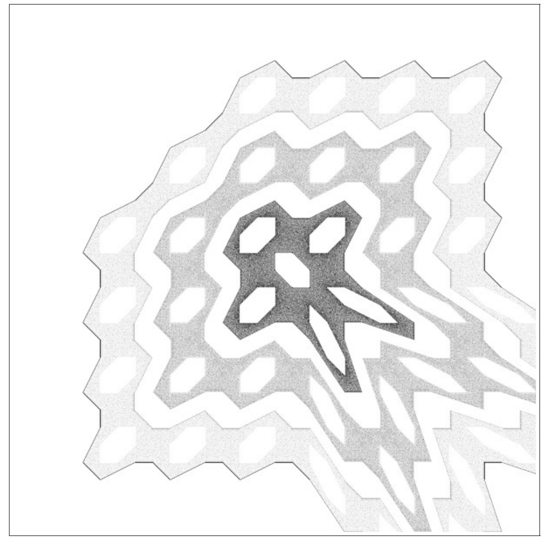
Here the square root is replaced by a sin function. The results are quite unexpected.

The Gingerbread Man

$x' = y + f(x)$ where $f(x) = \text{ABS}(bx)$
$y' = a - x$

The default parameters just generate 6 dots but try starting the iteration from a point just to the left of the origin. Alternatively, increase *b* just a little.

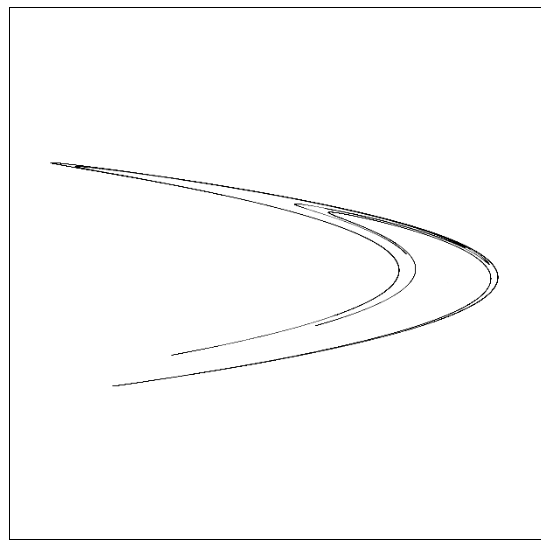
Unlike the classic Barry Martin fractals which appear to generate a single unique fractal which extends over the whole plane, the gingerbread function appears to be confined to a finite area. By clicking on the main window using the *right* hand mouse button you can *add* more fractals or periodic loops around him.



The Henon attractor

$x' = 1 + by + cx^2$
$y' = ax$

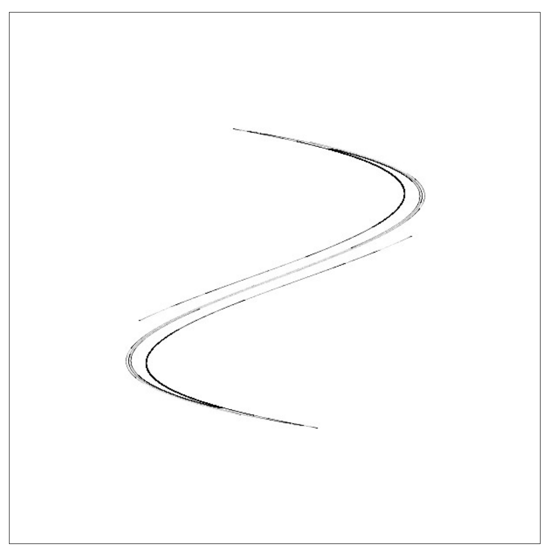
This is a proper attractor because the same pattern is generated wherever you start from (within a certain region) but it depends critically on the values of the parameters. The default parameters are (0.5, 0.6, 1.4). Another interesting set of parameters is (-1, 1, 0.5). This does not generate a strange attractor but produces a series of distorted circles depending on the initial starting point.



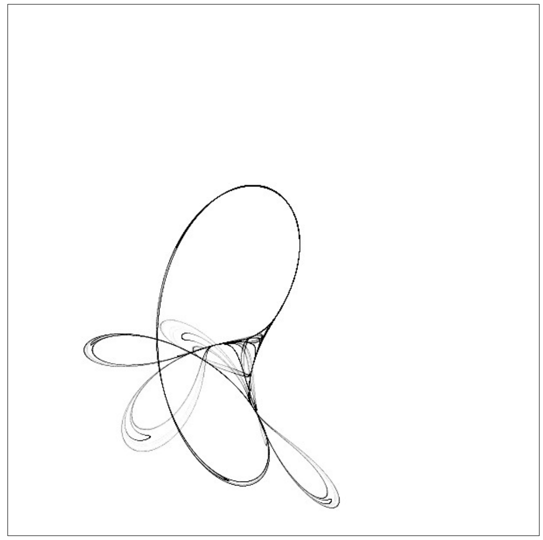
The Duffing attractor

$x' = ax - by - x^3$
$y' = x$

Like the Henon attractor, this attractor exists only for a small range of parameters.



The Tinkerbell attractor

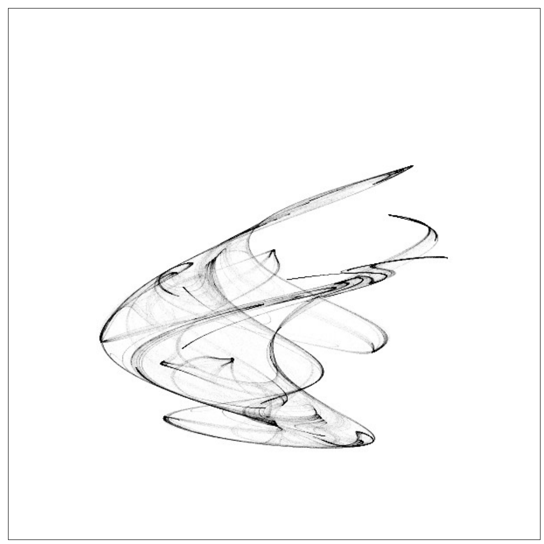


$$x' = (x^2 - y^2) + ax + by$$
$$y' = 2xy + cx + dy$$

This is a particularly attractive attractor and you can find several other sets of parameters which also work e.g. (0.9, -0.22, 2, 0.5) and (0, 0.15, 2, 0.65) etc. All of them have a pleasing 3 dimensional look but this is totally misleading especially if you are familiar with the Lorentz attractor which is a genuine 3D attractor. This is completely different. The points *do not* follow the apparent lines; they jump around all over the place. Consider one of the points where lines appear to cross (e.g. the point (-0.74, -0.44) in the default set). This point must go *somewhere* and it can only go to *one place*. It is impossible that a point "arriving from below" should "continue on upwards" or that one "arriving from the left" should "continue to the right". The attractor is simply a set of points on the 2 dimensional plane which are selected by the given parameters.

Now it may well be (and I strongly suspect that it is) that there exists a 3 dimensional algorithm which has a 3 dimensional attractor and that the Tinkerbell algorithm is a 2 dimensional projection of this attractor. If so this would explain its appearance nicely but I may be completely wrong about this.

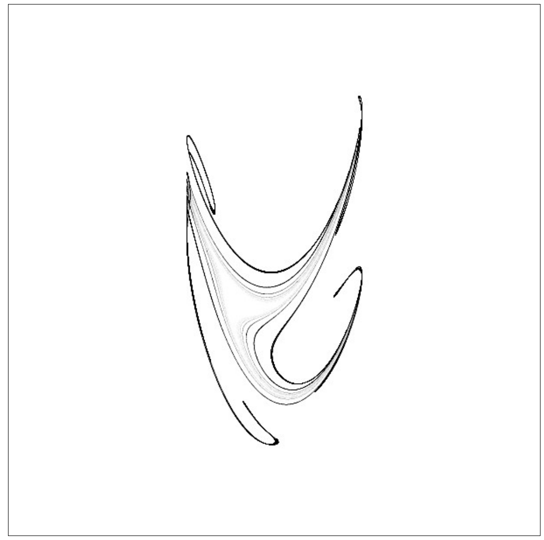
The de Jong attractor



$$x' = \sin(ay) - \cos(bx)$$
$$y' = \sin(cx) - \cos(dy)$$

It is immediately obvious that x and y cannot stray outside the box from (-2, -2) to (2, 2) whatever the values of a, b, c and d but, unlike most other algorithms, almost every set of parameters generates what looks like a strange attractor. I say 'looks like' because just running a computer program for a few million iterations does not guarantee that certain apparently blank areas of the plane will not eventually get filled.

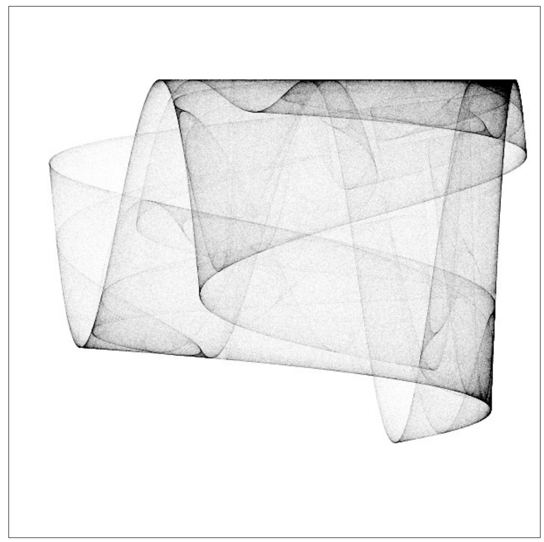
The Linton 'Ghost' attractor



$$x' = \sin(ay) - bx$$
$$y' = cx - \cos(dy)$$

I shall finish with two attractors of my own. The first is based on the de Jong algorithm and produces a wide variety of seemingly random squiggles, one of which looks quite like a ghost.

The Linton 'Tartan' attractor



$$x' = ay - b$$
$$y' = c - x^2$$

This algorithm generates a sort of 2 dimensional version of the well known logistic equation. You will see what I mean if you use the default values with b set to zero.

The reason why you get the tartan effect is as follows. If you calculate the second iterate (with $b = 0$) you get the following equations:

$$x'' = ay' = a(c - x^2)$$
$$y'' = c - x'^2 = c - ay^2$$

The significance of this is that x'' depends only on x and y'' depends only on y; and both equations are variations of the standard logistic equation $x' = \mathrm{A}x(1 - x)$.

Putting b equal to a small value skews the attractor giving it a pleasing 'linen fold' effect as shown in the illustration above.

Carr Bank: November 2016