Deterministic equations

Deterministic equations have the property that for every point in a multidimensional space, there exists unique vector. This vector can be used to determine a new point nearby which can, in turn, be used to specify a third point etc. In this way, a set of deterministic equations defines a possibly infinite set of trajectories through the space whose shapes and properties are by no means obvious from the superficial consideration of the original equations.

We can specify the vector field by defining a gradient with respect to an independent variable *t* (which we may think of as time if we wish) in each of the several dimensions as follows:

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = f(x, y, z...) \\ \frac{dy}{dt} &= \dot{y} = g(x, y, z...) \\ \frac{dz}{dt} &= \dot{z} = h(x, y, z...) \end{aligned} \qquad (1)$$

where *f*, *g* and *h* etc. are one-valued functions of *x*, *y* and *z*...

It is convenient to restrict ourselves to polynomial functions as these are easy to calculate and display more than enough interesting behaviour.

One dimensional equations

The generic one-dimensional deterministic equation (with positive exponents) is::

$$\dot{x} = a + bx + cx^2 + ... \qquad (2)$$

but if we are only interested in the qualitative behaviour of the system around its equilibrium points, we can restrict ourselves to those equations which cross the X axis at the maximum number of points. i.e.:

$$\dot{x} = \pm(x - a)(x - b)(x - c)... \qquad (3)$$

where *a*, *b*, *c* etc are all real. The simplest case (with *a* =0) is

$$\dot{x} = x \qquad (4)$$

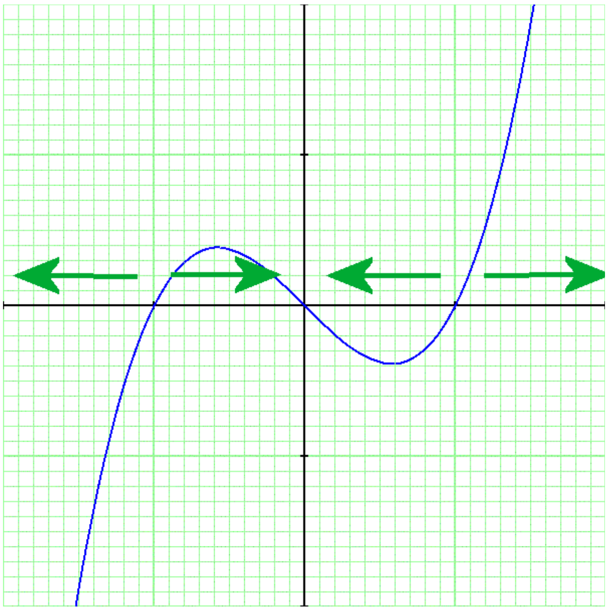
which has an equilibrium point at $x = 0$. Since positive values of *x* move in the positive direction and negative ones in the negative direction, this is an *unstable* equilibrium point or *source*.

On the other hand the equation

$$\dot{x} = -x \qquad (5)$$

is a sink:

In the general case (equation (2)) the line has alternate sinks and sources at points $x = a$, *b*, *c*, etc. This is because the curve wiggles alternately above and below the axis. Where the gradient at the points where it crosses the axis is positive, the point will be a source; where it is negative, it will be a sink as illustrated below for a cubic equation



If the curve is tangent to the axis, or if it has a point of inflexion there and the gradient is zero, the point is called a *stator*. Stators can be either attractive, e.g. $\dot{x} = -x^2$, repulsive e.g. $\dot{x} = x^2$ or attractive on one side and repulsive on the other e.g. $\dot{x} = x^3$ and $\dot{x} = 1/x$. We shall avoid equations with reciprocal terms in them because they behave pathologically in the region where x is close to zero.

Two dimensional equations

The generic two-dimensional linear equations are:

$$\begin{aligned} \dot{x} &= a + bx + cy \\ \dot{y} &= d + ex + dy \end{aligned} \tag{6}$$

By suitable changes of origin and scale, these can be reduced to:

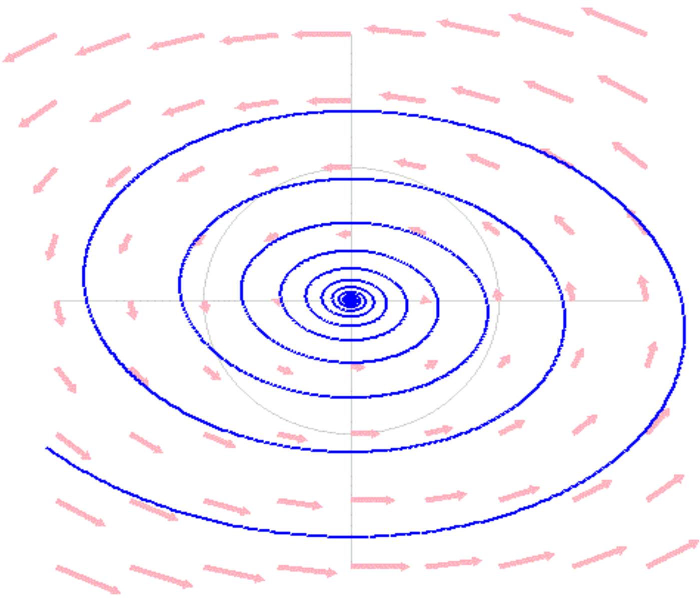
$$\begin{aligned} \dot{x} &= Ax + By \\ \dot{y} &= Cx + Dy \end{aligned} \tag{7}$$

Cyclic behaviour only results when $A = -D$ and B and C have opposite signs Circles only result when $A = D = 0$ and $B = -C$. The following table indicates the general type of behaviour that results

B&C \ A&D	**0 0**	**+ 0**	**0 –**	**+ +**	**+ –**
+ +	rectangular hyperbolas	hyperbolas	hyperbolas	hyperbolas	hyperbolas
+ –	circles	spiral out to infinity	spiral in to zero	spiral out to infinity	ellipses
– –	rectangular hyperbolas	hyperbolas	hyperbolas	hyperbolas	hyperbolas
+ **0**	straight line to infinity	straight line to infinity	straight line to axis	curve to infinity	hyperbolas

The elliptical spiral below was generated by the following equations:

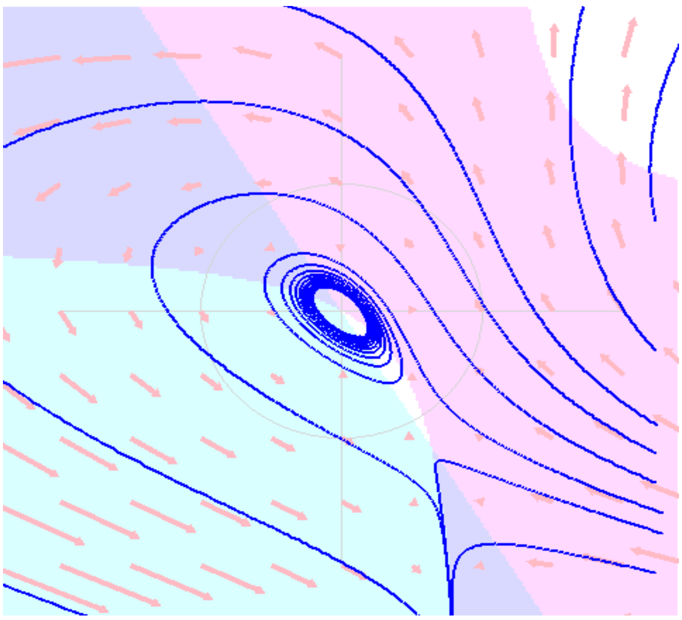
$$\begin{aligned}\dot{x} &= -0.2\,x - 2\,y \\ \dot{y} &= x\end{aligned} \qquad (8)$$



One interesting theorem is that if any point in the plane generates a cyclic trajectory, then all points will. The reason is that, because both equations are linear, all patterns obtained must be independent of scale. If you multiply *x* and *y* by a constant factor, then $\dot{x}$ and $\dot{y}$ are multiplied by the same factor – i.e. the point will move in the same direction as before.

If we introduce a quadratic term such as *xy* (in one equation only) the same basic behaviour is exhibited but the curves are distorted in curious ways. For example, the following curve was obtained from the equations:

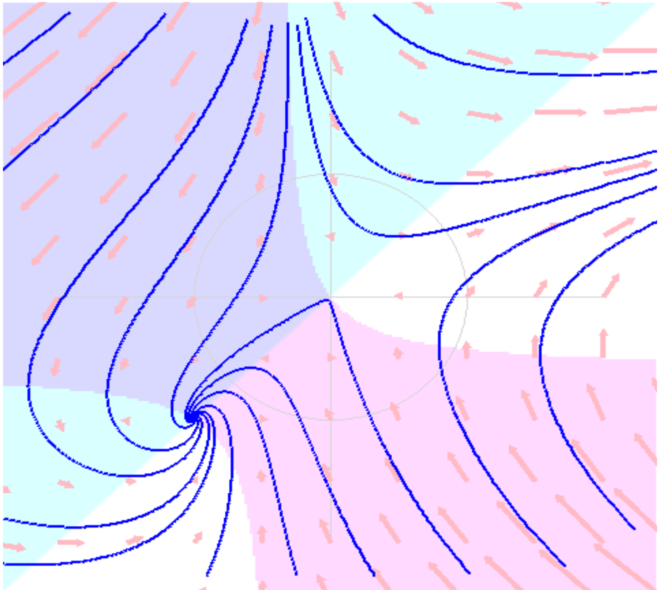
$$\begin{aligned}\dot{x} &= -0.6\,x - y + x\,y \\ \dot{y} &= x + 0.6\,y\end{aligned} \qquad (9)$$



By putting $\dot{x}$ and $\dot{y}$ equal to zero and solving it is easy to show that the system has two equilibrium points and these are obvious from the coloured map of the function. (points where $\dot{x}$ is negative are shaded in blue, points where $\dot{y}$ is negative are red.) It is clear from the way that the curve spirals in that the origin is a stable equilibrium point. The other one is unstable.

Here is another rather pretty example with

$$\begin{aligned} \dot{x} &= 0.4\,x + 0.4\,y + x\,y \\ \dot{y} &= x - y \end{aligned} \tag{10}$$



If you change the signs of all the parameters, all the arrows simply reverse. What this means is that the sink (at (-1.2, -1.2) in the image above) turns into a source. The two-dimensional case introduces a third kind of equilibrium point called a *saddle* point which is attractive in one direction but repulsive in another. Since a quadratic system can have, at most, 2 equilibrium points there are 6 possibilities

A. 1 sink, 1 saddle
B. 1 source, 1 saddle
C. 1 sink, 1 source
D. 2 sinks
E. 2 sources
F. 2 saddles

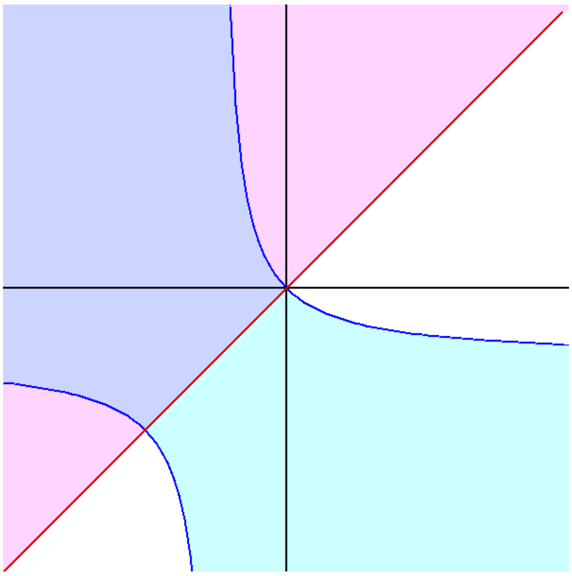
We have seen cases A and B but what about the others?

Classifying equilibrium points

The best way to make sense of these patterns is to consider the points on the plane at which $\dot{x}$ and $\dot{y}$ are positive, negative or zero. In the case of the example given above, we need to consider the two equations

$$\begin{aligned} 0.4\,x + 0.4\,y + x\,y &= 0 \\ x - y &= 0 \end{aligned} \tag{11}$$

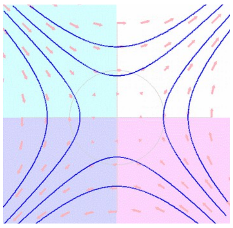
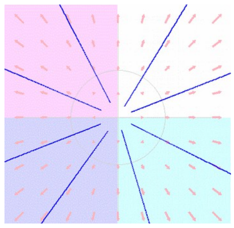
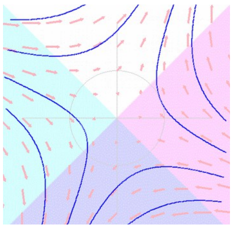
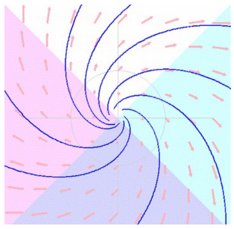
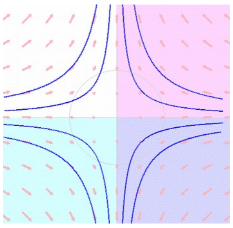
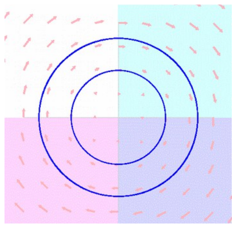
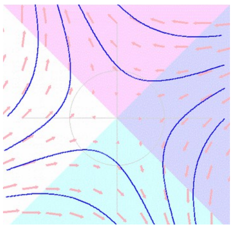
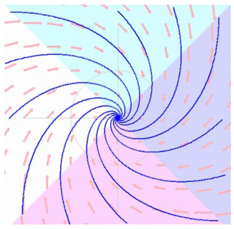
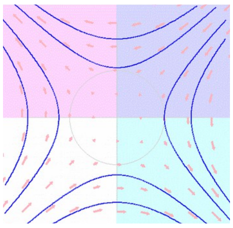
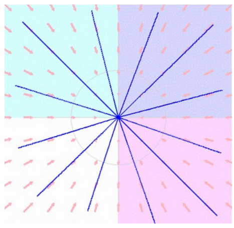
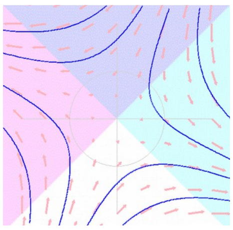
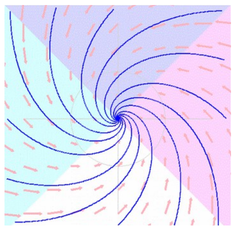
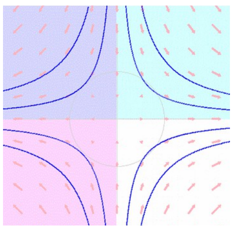
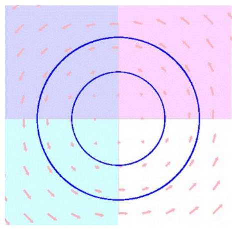
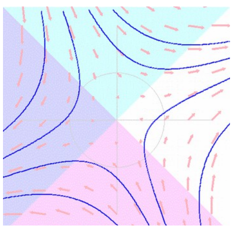
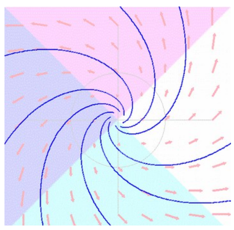
In the diagram below, the shaded regions are where the variables $\dot{x}$ (in blue) and $\dot{y}$ (in red) are negative.



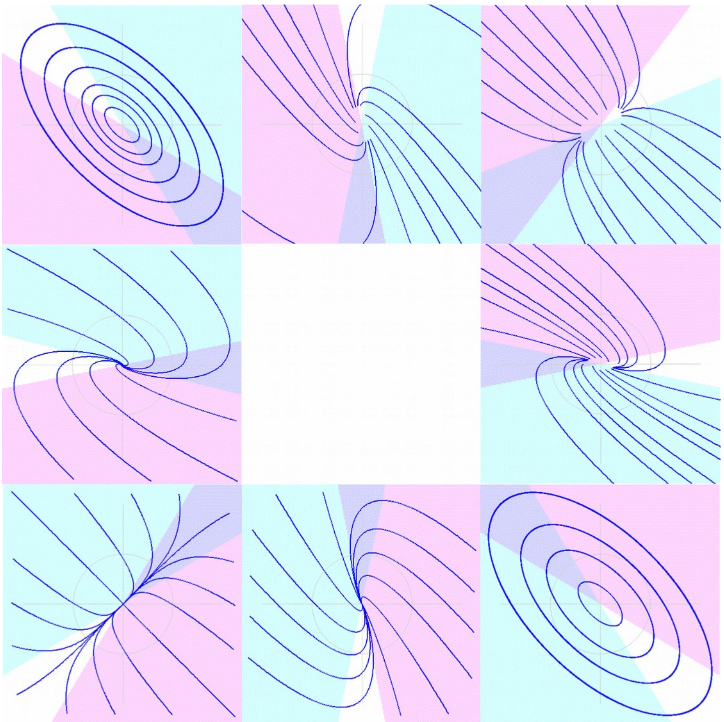
If you look at the two equilibrium points in the above example, you will see that they have different handedness. At the origin, in a clockwise direction we have white > pink > blue > cyan while at the other one we have white > cyan > blue > pink. What this means in practice is that, while one variable will have the same gradient at both points, the other will have opposite gradients. This means that one of the points (and only one) *must* be a saddle. Cases C, D, E and F are all impossible.

When it comes to identifying the different types of equilibrium point, it is worth bearing in mind that it is only the linear terms which actually matter. (By a suitable change in coordinates, any equilibrium point can be placed at the origin where quadratic and higher order terms become vanishingly small.) The following table lists all the possible cases linear cases where the two functions listed in equation (7) cross at right angles.

The table is listed in order with the quadrant in which both $\dot{x}$ and $\dot{y}$ are positive moving round anti-clockwise in steps of 45°. In the right hand column, $\dot{x}$ and $\dot{y}$ are reversed thus reversing the handedness of the map. (A and B are swapped with C and D). It is worth noting that all the left-handed maps produce saddle points; the right-handed maps generate sources, sinks and cyclic points. One interesting thing to note about the hyperbolas generated by the left-handed maps is that in one complete rotation of the map, the hyperbolas rotate by 180°.

A	B	C	D	Map	A	B	C	D	Map
0	1	1	0		1	0	0	1	
-1	1	1	1		1	1	-1	1	
-1	0	0	1		0	1	-1	0	
-1	-1	-1	1		-1	1	-1	-1	
0	-1	-1	0		-1	0	0	-1	
1	-1	-1	-1		-1	-1	1	-1	
1	0	0	-1		0	-1	1	0	
1	1	1	-1		1	-1	1	1	

If the $\dot{x} = 0$ and $\dot{y} = 0$ lines make a smaller (or larger) angle, the twisted spirals are distorted and the circles become ellipses:



(The left handed versions are, of course, all saddles.)

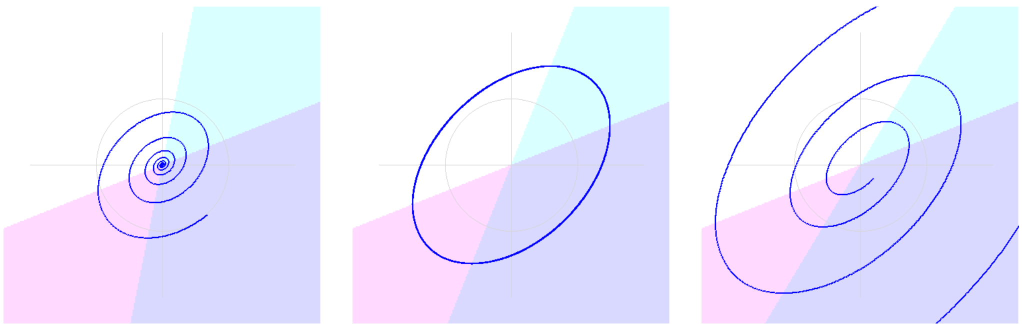
Predicting the behaviour of equilibrium points

How can we predict from the equations which kind of equilibrium point we will get?

Consider the following three very similar equations which actually behave very differently:

$$\begin{aligned} \dot{x} &= -0.4x + y \\ \dot{y} &= -x + 0.2y \end{aligned} \qquad \begin{aligned} \dot{x} &= -0.4x + y \\ \dot{y} &= -x + 0.4y \end{aligned} \qquad \begin{aligned} \dot{x} &= -0.4x + y \\ \dot{y} &= -x + 0.6y \end{aligned}$$

This is how they behave:



Spiral in Cycle Spiral out

Clearly there is something special about the middle case. As we have seen, in order to generate a cyclic orbit, *A* must equal -*D* and *B* and *C* must have opposite signs. Why is this?

I think we can assume that symmetry dictates that the only cyclic orbits are ellipses (centred on the origin). The equation of such an ellipse is

$$ax^2 + by^2 + cxy = R \tag{12}$$

where *a*, *b* and *R* are positive numbers. (*c* can be either sign.)

By differentiating we find that

$$\frac{dx}{dy} = -\frac{cx + 2by}{2ax + cy} \tag{13}$$

This function is consistent with the condition that

$$\begin{aligned} \dot{x} &= -cx - 2by \\ \dot{y} &= 2ax + cy \end{aligned} \tag{14}$$

(or its negation).

If you compare this with equation (6) you will see that these equations imply that $A = -c$, $B = -2b$, C = 2a and D = c. In other words, A must equal -D and B and C must have opposite signs.

But these are not quite the only conditions which have to be satisfied. We have shown that, if the orbit is an ellipse, then these conditions must apply. We have not shown that when these conditions apply, the orbit will always be an ellipse. We must also ensure that the equilibrium point has the correct *handedness*.

A bit of experimenting leads us to the conclusion that, for cyclic orbits to occur

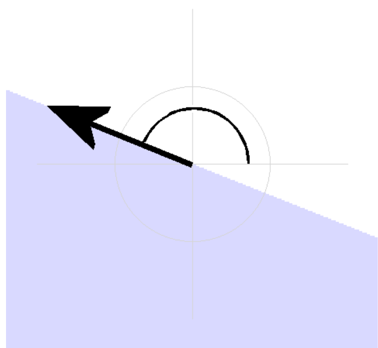
$$|B \times C| > |A \times D| = A^2 \tag{15}$$

(Since both pairs have opposite signs, it is best to ignore the signs completely)

If $|B \times C| = |A \times D|$ the ellipses degenerate into a series of parallel lines and if $|B \times C| < |A \times D|$ the ellipses become hyperbolas and the equilibrium point becomes a saddle.

The next point of interest is what happens if A and D are not quite (numerically) equal? Instead of being a stable cycle, the point either spirals in to the centre or out to infinity. To discover the condition which determines which possibility occurs, let us first look at the difference between sources and sinks.

Since, in general A and D are not equal, we must use the most general form of the handedness condition to separate sources and sinks from saddles. The crucial aspect is the relation between the gradients of the lines $\dot{x} = 0$ and $\dot{y} = 0$ which are $-A / D$ and $-D / C$ respectively. But it is not just the magnitude which matters because we must distinguish between lines which have positive values lying on the right from lines which have positive values on the left. What we need, in fact, is the *arctangent* of the gradient. Now there are 4 quadrants to consider depending on the signs of A and D (or B and C). Let us define the positive direction of a line as one in which negative values of x (or y) are on the left and positive values on the right. Take the case of $A = 0.4$ and $B = 1$ shown below.



The gradient of the line is -0.4 but the arctangent lies in the second quadrant (the shaded area represents negative values.) Now we do not have to actually calculate the arctangents because all we are interested in here is their relative sizes. We can tabulate the possibilities as follows: (where an condition is specified, the result could be a source or a sink if the condition is satisfied, otherwise it is a saddle.)

$\dot{x} = 0 \backslash \dot{y} = 0$	Quad 1 (C+, D-)	Quad 2 (C+, D+)	Quad 3 (C-, D+)	Quad 4 (C-, D-)
Quad 1 (A+, B-)	$\lvert AD\rvert < \lvert BC\rvert$	Anti-clockwise Source	$\lvert AD\rvert > \lvert BC\rvert$	Saddle
Quad 2 (A+, B+)	Saddle	$\lvert AD\rvert > \lvert BC\rvert$	Clockwise Source	$\lvert AD\rvert < \lvert BC\rvert$
Quad 3 (A-, B+)	$\lvert AD\rvert > \lvert BC\rvert$	Saddle	$\lvert AD\rvert < \lvert BC\rvert$	Clockwise Sink
Quad 4 (A-, B-)	Anti-clockwise Sink	$\lvert AD\rvert < \lvert BC\rvert$	Saddle	$\lvert AD\rvert > \lvert BC\rvert$

Although there is a great deal of symmetry in this table, there appears to be no simple formula in *A*, *B*, *C* and *D* which will immediately tell you what kind of equilibrium point you will get. It is, however, easy enough to use the table either to predict the behaviour of a set of equations or to construct a set with given behaviour.

Constructing quadratic maps

For example, suppose we want to construct a set of equations which has a clockwise source at the origin and a saddle at (1,1). We know immediately that the $\dot{x} = 0$ line at the origin must be in the second quadrant and that therefore the $\dot{y} = 0$ line at the origin must be in either quadrants 2, 3 or 4. Now since the saddle point at (1,1) must also lie on this line, the line must, in fact be:

$$\dot{y} = -x + y \tag{16}$$

A suitable function for $\dot{x}$ is:

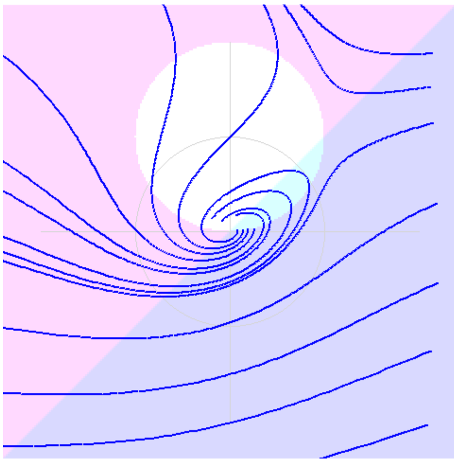
$$\dot{x} = -x^2 - y^2 + 2y \tag{17}$$

At the origin, we can throw out all the quadratic terms leaving just

$$\dot{x} = 2y \tag{18}$$

This will generate a clockwise source because *A* and *B* are both positive (or zero in the case of the former).

The map looks like this:

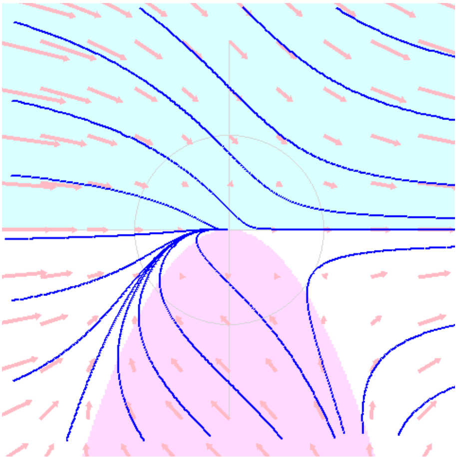


Degenerate points

In addition, there are, degenerate cases where the $\dot{x} = 0$ and $\dot{y} = 0$ curves are tangential. An example is the following:

$$\begin{aligned} \dot{x} &= x^2 + y \\ \dot{y} &= -y \end{aligned} \tag{19}$$

where the sink and the saddle merge at the origin. This is what it looks like:



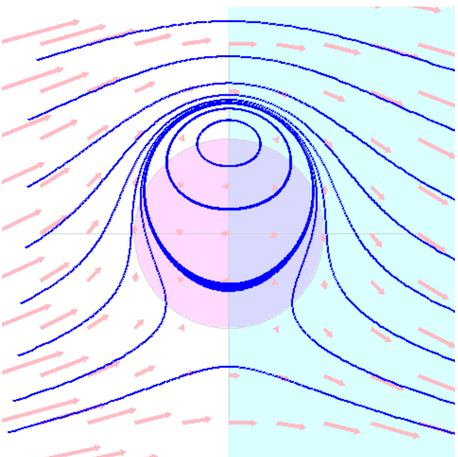
This is a *stator*. Points which approach from the left get slower and slower and never reach the origin. Points which start close to the origin with small but positive x coordinates get propelled faster and faster along the X axis like bullets from a gun.

Circular quadratic equations

Some particularly interesting maps are produced when the quadratic equation is circular. For example, the equations:

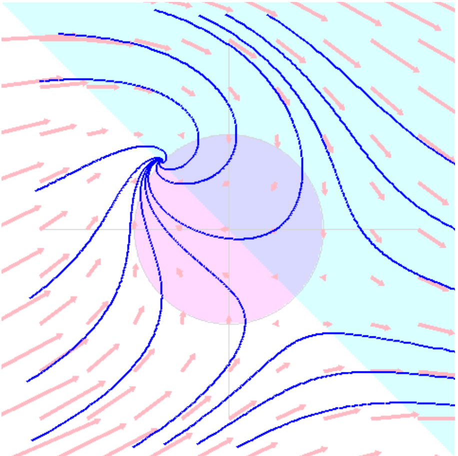
$$\begin{aligned} \dot{x} &= x^2 + y^2 - 1 \\ \dot{y} &= -x \end{aligned} \tag{20}$$

generates this:



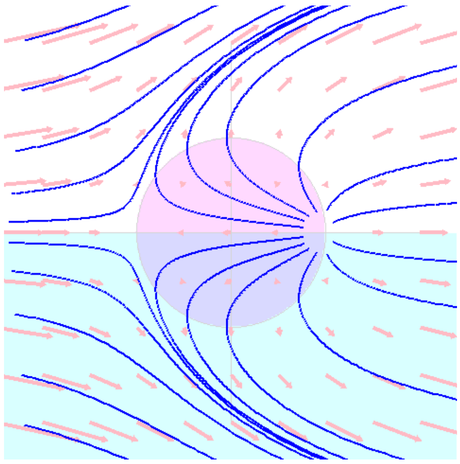
As you can see, the equilibrium point at (0,1) is cyclic because the $\dot{x} = 0$ and $\dot{y} = 0$ curves cross parallel with the axes. (Many of these maps remind me of possible flow patterns in fluids but there is something wrong with the parallel. For example, the streamlines just above the point (0,1) are far too close together.)

Turning the $\dot{y}$ curve by 45° produces this:



The cyclic point has become a twisted sink. Further revolutions of the linear curve generate first a straight sink, then a twisted sink, then a cyclic point rotating in the opposite direction followed by a series of twisted and straight sources. All the while, the saddle point swings round at the opposite end of the diameter.

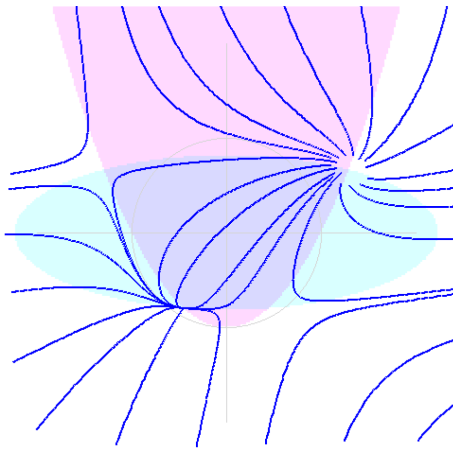
Turning the linear equation the other way produces this:



which reminds me of the electric field round a positive charge placed in a uniform field.

Double quadratic functions

So far we have only considered cases in which the $\dot{y}$ equation is linear. If it, too, is quadratic, then in general we can expect four equilibrium points as in the example below:



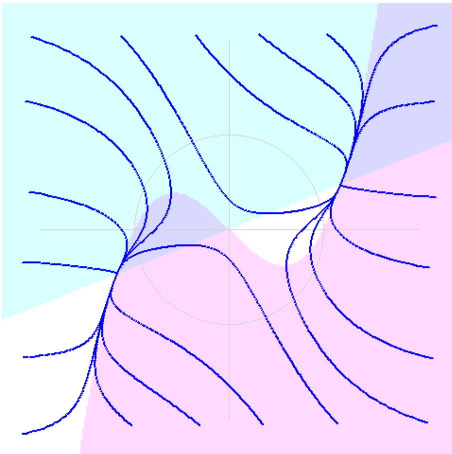
It will be seen that we have a source, a sink and two saddles. I do not have a proof but I do not believe that any other combination is possible.

Cubic equations

If we permit cubic equations in the function for $\dot{x}$, the possible sets of equations become enormous – but a few simple example will set the pattern for the rest. Obviously the first thing to say is that , in general, there will be three equilibrium points – but what combinations are possible? Consider one of the simplest functions:

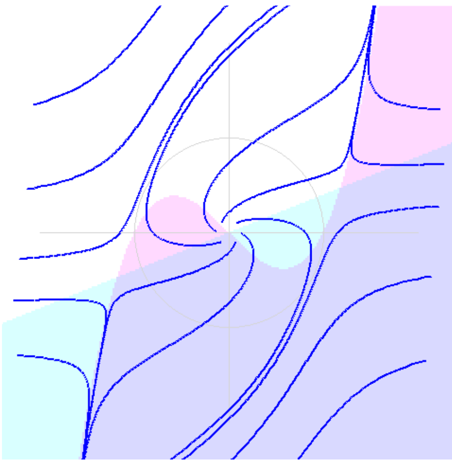
$$\begin{aligned} \dot{x} &= x + y - x^3 \\ \dot{y} &= 0.4\,x - y \end{aligned} \tag{21}$$

which has the shape:



with two sinks and a saddle point in between. It is pretty obvious that, traced along the line of either equation, the equilibrium points will be alternately either sinks and saddles or sources and saddles and that this scheme will extend up to equations of any order. (It may be wondered why it is possible to obtain multiple sinks from a combination of a linear and a cubic equation but it is, apparently, impossible to obtain two sinks from a combination of two quadratics. I believe the answer lies in the way two quadratics have to curl round each other in order to intersect four times.)

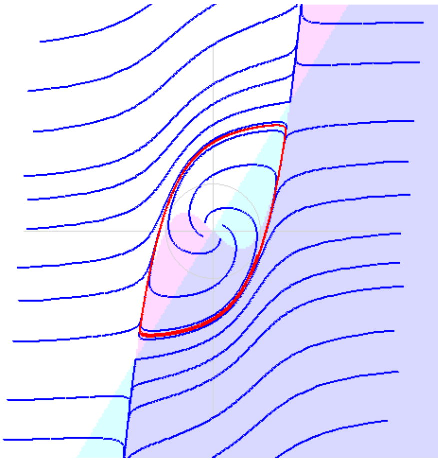
If we reverse the handedness of the above example by putting $\dot{y} = y$ we get one source and two saddles:



Now rotate the linear equation anti-clockwise and something very remarkable and quite unexpected happens: (Here are the equations:

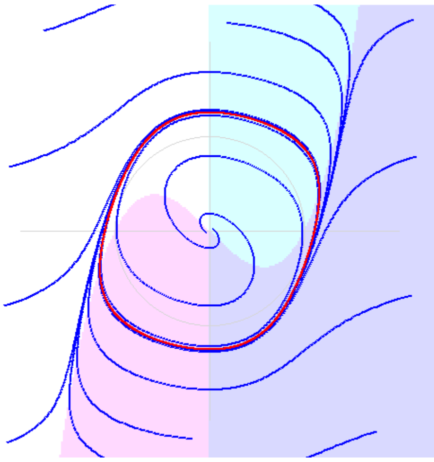
$$\begin{aligned} \dot{x} &= x + y - x^3 \\ \dot{y} &= -x + 0.6\,y \end{aligned} \tag{22}$$

The points which spiral out from the central source get caught up in a continuous loop – shown in red. So do many points which come in from outside. This set of points is a *cyclic attractor.* Obviously, if a function contains a cyclic attractor, it divides the plane into two regions. Inside the sink there will be a single source with two saddle points outside.



If the map is reversed, the source becomes a sink and the loop becomes a *cyclic repellor.* (i.e. any point exactly on the loop will cycle indefinitely but the slightest deviation and it will spiral away.)

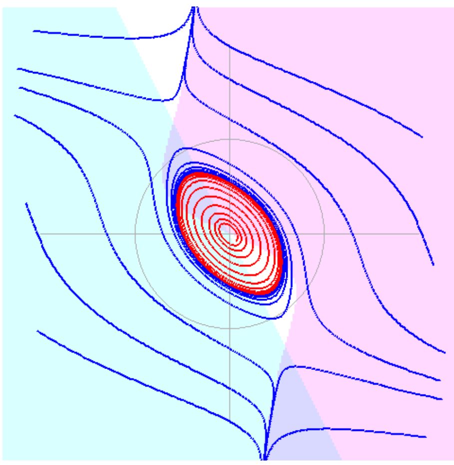
If you continue to rotate the linear equation to the point where it lies along the Y axis, *all* points in the plane become attracted by the attractor.



Further anticlockwise rotation reduces the attractor until eventually it splits into two point sinks separated by a saddle.

Cyclic attractors are not easy to find. Here is another one using the following equations:

$$\begin{aligned} \dot{x} &= -0.4\,x - 0.9\,y - xy^2 \\ \dot{y} &= x + 0.5\,y \end{aligned} \tag{23}$$

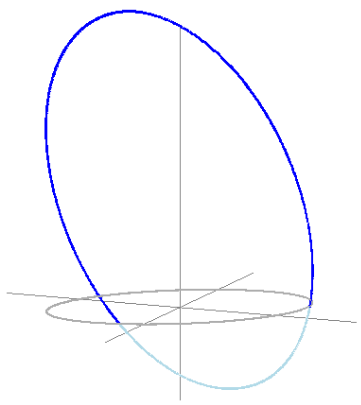


Three dimensional equations

With three dimensions to play with, we move into a whole new ball game with a huge number of possible equations to consider. If we start with the linear case, we still have 9 coefficients to deal with:

$$\begin{aligned} \dot{x} &= A_x\,x + B_x\,y + C_x\,z \\ \dot{y} &= A_y\,x + B_y\,y + C_y\,z \\ \dot{z} &= A_z\,x + B_z\,y + C_z\,z \end{aligned} \tag{24}$$

but most of the time, the trajectories will rapidly disappear off to infinity in one direction or another. If we are interested in finding trajectories which are cyclic, we can start with cases where $A_x = -B_y$ and B_x and A_y have opposite sign as in the 2 dimensional case. This leads us to cases like the following:

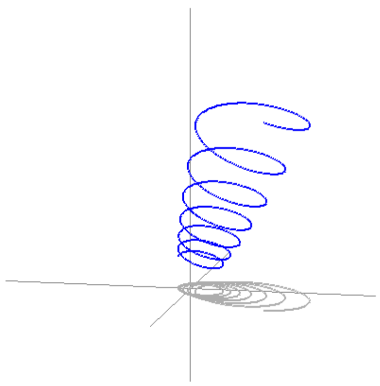


$$\begin{aligned} \dot{x} &= 0.4\,x - y \\ \dot{y} &= 0.8\,x - 0.4\,y \\ \dot{z} &= 0.5\,x + y \end{aligned} \tag{25}$$

Provided $A_x = -B_y$ and the C terms are all zero, the result is always an inclined ellipse.

If $A_x \neq -B_y$ then the point spirals in or out as inj the 2D case but if the C terms are all zero, the spiral remains in one plane.

As soon as you start tinkering with any of the C terms, the ellipse or spiral typically becomes a squashed helix as shown below.



Introducing a quadratic term opens up an infinitude of possibilities.

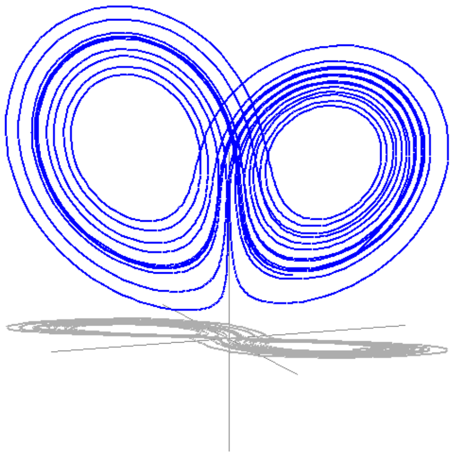
The famous Lorentz 'butterfly' attractor is traditionally generated from the following equations:

$$\dot{x} = -10\ x + 10\ y$$
$$\dot{y} = 28\ x - y - xz$$
$$\dot{z} = -8/3\ z$$

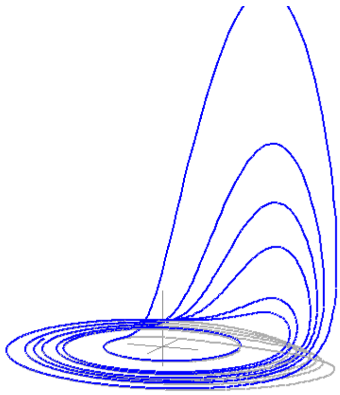
but by scaling the coefficients down and simplifying them without losing the desired behaviour we can use the simpler set:

$$\dot{x} = -x + y$$
$$\dot{y} = 2\ x - xz$$
$$\dot{z} = -0.3\ z$$

which looks like this:



Another attractor which has only one quadratic term is the Rossler attractor:



whose equations are:

$$\dot{x} = -y - z$$
$$\dot{y} = x + 0.2\ y$$
$$\dot{z} = 0.2 - 5.7\ z + xy$$

The linear terms in x and y establish an outgoing spiral in the $x\,y$ plane. The positive and negative terms in the equation for $\dot{z}$ cause the point to rise and fall as it goes round and the y term in the $\dot{y}$ equation ensures that when the point rejoins the outgoing spiral, it does so almost at random. This value is critical.