Bernoulli Numbers

Introduction

In the article on Pascal's Triangle, the following results were obtained

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n.(n + 1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{i=1}^{n} i^2 = \frac{n.(n+1).(2n + 1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n.(n+1)}{2}\right)^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$\sum_{i=1}^{n} i^4 = \frac{n.(n + 1).(2n + 1).(3n^2 + 3n - 1)}{30}$$
$$= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$\sum_{i=1}^{n} i^5 = \frac{n.n.(n + 1).(n + 1)(2n^2 + 2n - 1)}{12}$$
$$= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

In general, $\sum_{i=1}^{n} i^s$ is a polynomial expression in n of degree $s + 1$.

Lets compare these coefficients with the well known binomial coefficients.

s+1	Binomial coefficient P	Series coefficient S	$(s + 1) \times S / P$
1	1 : 1	1	1
2	1 : 2 : 1	1/2 :1/2	1 : 1/2
3	1 : 3 : 3 : 1	1/3 : 1/2 : 1/6	1 : 1/2 : 1/6
4	1 : 4 : 6 : 4 : 1	1/4 : 1/2 :1/4 : 0	1 : 1/2 : 1/6 : 0
5	1 : 5 :10 :10 : 5 : 1	1/5 : 1/2 : 1/3 : 0 : −1/30	1 : 1/2 : 1/6 : 0 : −1/30
6	1 : 6 :15 : 20 :15 : 6 : 1	1/6 : 1/2 : 5/12 : 0 : −1/12 : 0	1 : 1/2 : 1/6 : 0 : −1/30 : 0

Remarkably we see a pattern emerging.

The coefficient of the k^{th} term (numbered from 0 in order of *decreasing* power) S_k is related to the equivalent Binomial coefficient P_k $\left(= C_k^{s+1} = \frac{(s + 1)!}{k!(s+1-k)!}\right)$ by the following relation:

$$(s + 1)\frac{S_k}{P_k} = B_k \qquad (1)$$

where B_k is a constant whose values are (numbered from zero):

$$1,\ 1/2,\ 1/6,\ 0,\ -1/30,\ 0,\ 1/42,\ 0,\ -1/30, \text{ etc}$$

We noted in the other article that when $s = 6$ ($s + 1 = 7$) a factor of 7 had to appear in the numerator so the fact that $B_7 = 1/42$ is no surprise. Indeed we should expect to see the prime numbers playing an important role in the determination of the Bernoulli numbers.

(We shall accept that the pattern continues indefinitely without proof!)

The question now arises, can we find a formula for the Bernoulli numbers?

Recursive definition

Rearranging equation (1) we have:

$$S_k = \frac{P_k B_k}{s + 1}$$

which means that taking, for example, the third term of the expansion of $\sum_{i=1}^{n} i^4$ (ie $k = 2$)

$$S = \frac{P_2 B_2}{4 + 1} = \frac{C_2^5 B_2}{5} = \frac{10 \times 1/6}{5} = \frac{1}{3}$$

and
$$\sum_{i=1}^{n} i^4 = \frac{C_0^5 B_0}{5} n^5 + \frac{C_1^5 B_1}{5} n^4 + \frac{C_2^5 B_2}{5} n^3 + \frac{C_3^5 B_3}{5} n^2 + \frac{C_4^5 B_4}{5} n$$

so

$$\sum_{i=1}^{n} i^s = \frac{1}{s+1} \sum_{j=0}^{s} C_j^{s+1} B_j n^{s+1-j}$$

NB the sum goes from 0 to s not $s + 1$ because the there is never any unit term.

If we put $n = 1$ we get

$$\sum_{i=1}^{1} i^s = 1 = \frac{1}{s+1} \sum_{j=0}^{s} C_j^{s+1} B_j$$

eg for $s = 4$ again

$$\frac{1}{5}\left(C_0^5 B_0 + C_1^5 B_1 + C_2^5 B_2 + C_3^5 B_3 + C_4^5 B_4\right)$$
$$= \frac{1}{5}\left(1.1 + 5.\frac{1}{2} + 10.\frac{1}{6} + 10 \times 0 + 5.\left(\frac{-1}{30}\right)\right) = 1$$

We can split off the last term like this:

$$1 = \frac{1}{s+1}\left(\sum_{j=0}^{s-1} C_j^{s+1} B_j + C_s^{s+1} B_s\right)$$

but since $C_s^{s+1} = s + 1$ we can say that:

$$(s + 1) B_s = s + 1 - \sum_{j=0}^{s-1} C_j^{s+1} B_j$$

or
$$B_s = 1 - \frac{1}{s + 1} \sum_{j=0}^{s-1} C_j^{s+1} B_j \qquad (2)$$

For example:

$$B_4 = 1 - \frac{1}{5}\sum_{j=0}^{3} C_j^5 B_j$$

$$B_4 = 1 - \frac{1}{5}\left(1B_0 - 5B_1 - 10B_2 - 10B_3\right)$$

$$B_4 = 1 - \frac{1}{5}.1.1 - \frac{1}{5}.5.\frac{1}{2} - \frac{1}{5}.10.\frac{1}{6} - 0 = -\frac{1}{30}$$

A simple algorithm based on equation (2) for calculating Bernoulli numbers is given below.

```
Private Function Bernoulli(ByVal n As Integer) As Double
   Dim A(n) As Double
   For m = 0 To n
     A(m) = 1 / (m + 1)
     For j = m To 1 Step -1
       A(j - 1) = j * (A(j - 1) - A(j))
     Next
   Next
   Return A(0)
 End Function
```

It turns out that all the odd Bernoulli numbers (>2) are zero and that alternate even numbers are positive and negative. All the numbers are rational fractions but they start to increase quite rapidly.

Bernoulli's numbers and the zeta function

Euler proved the quite astounding result that

$$\zeta(2n) = \frac{(4\pi^2)^n |B_{2n}|}{2(2n)!}$$

Putting $n = 1$, we get

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{4\pi^2 \times 1/6}{4} = \frac{\pi^2}{6}$$

and $n = 2$, we get

$$\zeta(4) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots = \frac{(4\pi^2)^2 \times 1/30}{48} = \frac{\pi^4}{90}$$