The Mandelbrot Map

To my dear wife

Carr Bank: February 2017, October 2022


The
Mandelbrot
Map
Oliver Linton

Julia sets

To generate a Julia set you start with a complex number $\mathbf{z}_0$ (the 'seed') and apply an iterative formula to it over and over again until the point either escapes to infinity or meets some other condition.

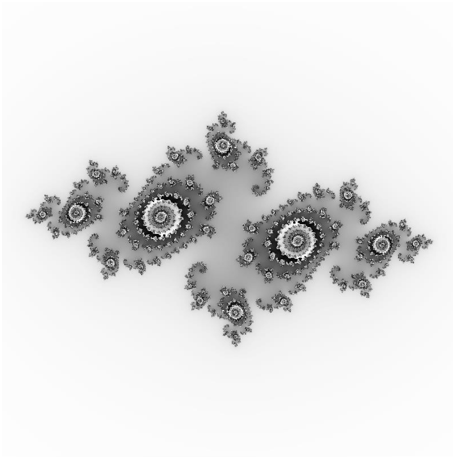
The classic formula which generates the famous Mandelbrot set is $\mathbf{z'} \Rightarrow \mathbf{z}^2 + \mathbf{C}$ and every value of **C** generates a different Julia set. A couple of examples are shown opposite.

Some Julia sets, like the one at the top, are made of separate pieces and the point in the middle (which corresponds to a seed of (0, 0)) escapes to infinity more or less quickly. In contrast, other sets like the one at the bottom are connected and the origin is stable.

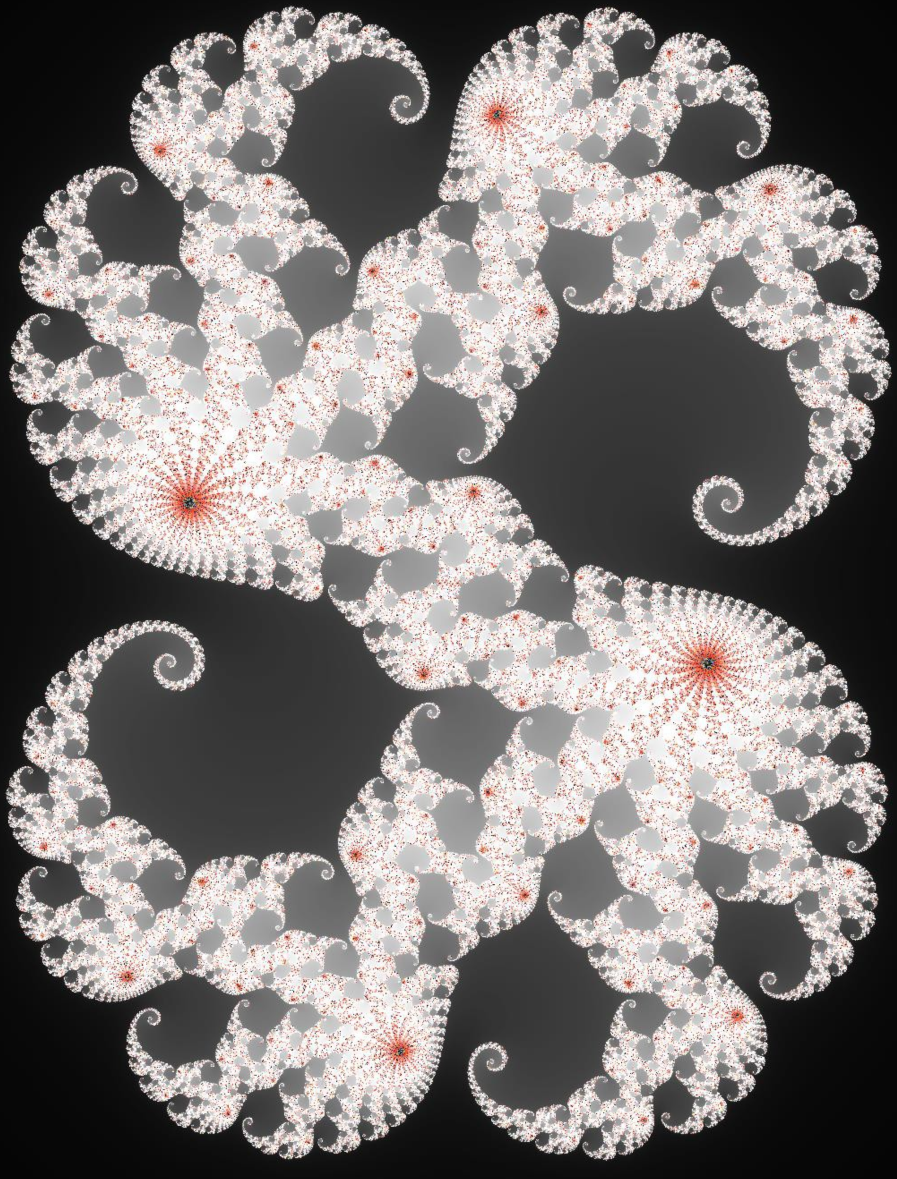
Julia sets can be very beautiful because of the high degree of symmetry which they display but although they contain infinite detail, magnifying them always reveals exactly the same sort of structure. They are described as being 'self-similar'.

The Mandelbrot Set is a kind of map of all Julia sets. It is generated by iterating a fixed seed ((0, 0) in the classic case) over all values of **C**. What this means is that the colour of the point on the Mandelbrot map is the same as the colour of the origin of the Julia set which corresponds to that value of **C**.

Unlike the Julia sets, the Mandelbrot Map is not self-similar and magnifying portions of it produces an endless series of subtly different structures.

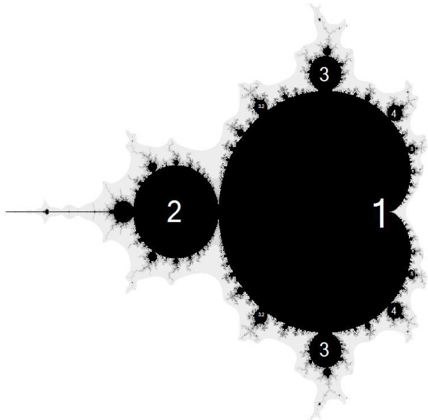


Elephant Julia set

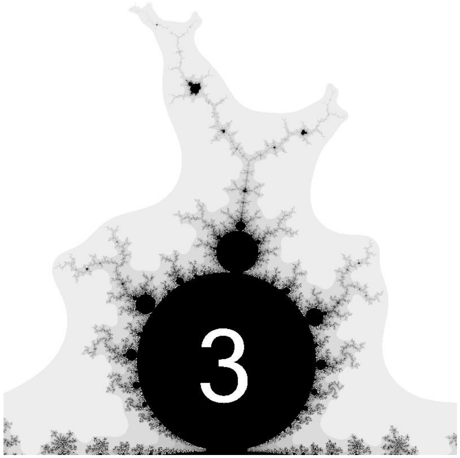


The Mandelbrot Map

The Mandelbrot Map is possibly the most intricate mathematical object known. It is so complex that zooming in to quite modest levels will get you to places that no one has ever seen before. It is also very easy to get lost. What we need is some way of identifying the different regions of the map. Fortunately, the map is highly organised and with a bit of practice it is possible to look at a certain small region and have a pretty good idea of where it is on the map. Lets see how the map can be labelled.

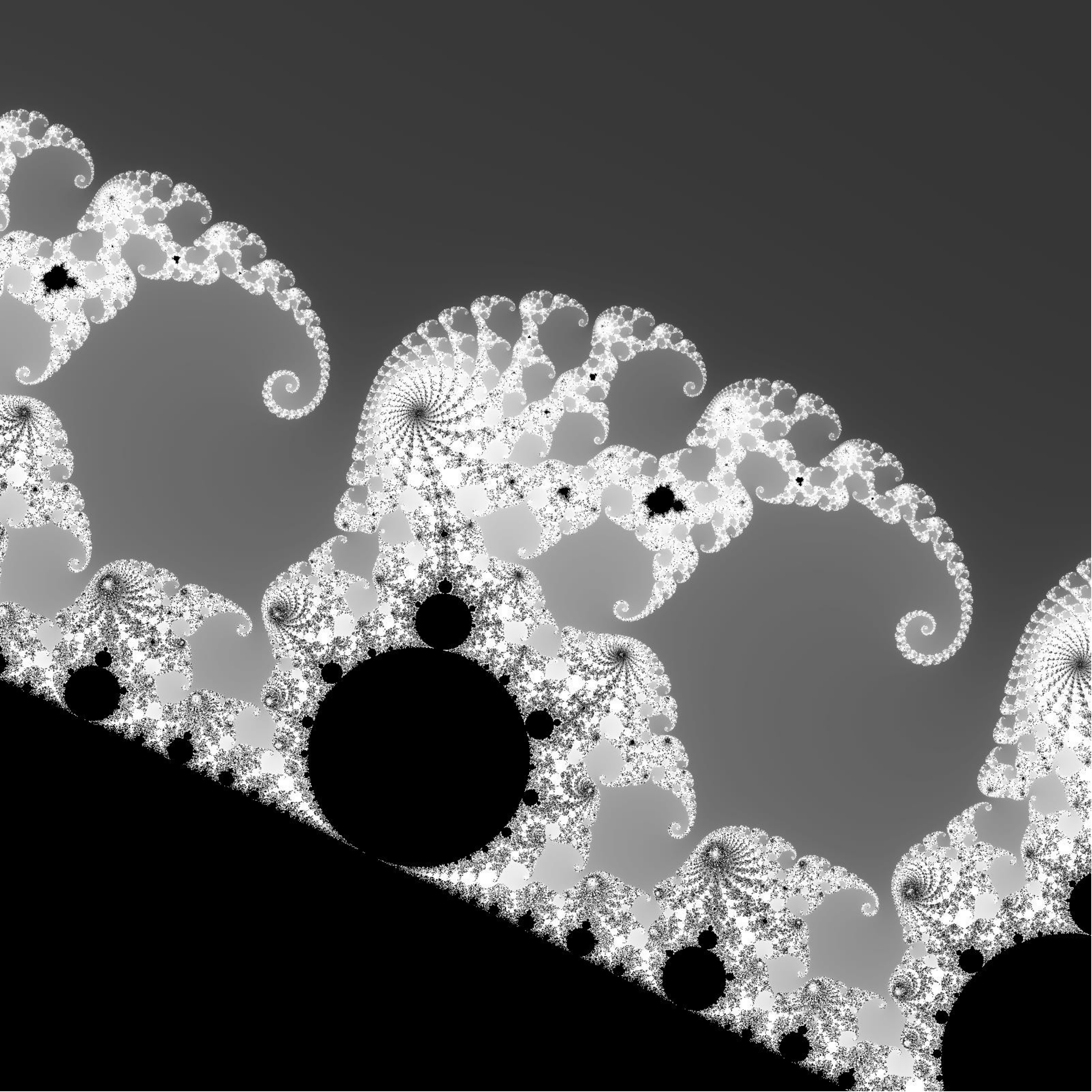


The most obvious feature of the map is that it consists of a lot of approximately circular 'blobs' stuck on other larger 'blobs'. Since the word 'blob' is rather vulgar, I shall refer to all the 'blobs' as **lobes.** The biggest lobe (which actually looks more like a cardioid) is lobe number 1. The next largest lobe (the 'frontal' lobe) is lobe number 2 and the two next largest lobes above and below (the 'temporal' lobes) are lobes number 3. (Since the map is symmetrical about the X axis, we shall just consider the upper lobes from now on.)



Now if you look closely at the main 'sprout' on lobe number 3, you will see that it soon splits into two branches at a major 'junction' which therefore has a total of 3 branches. This is a good reason for calling this lobe number 3. I shall refer to the main 'sprout' which connects to the lobe as an **axon**; I shall call the main junction a **synapse,** the branches **dendrites** and the whole assembly a **neuron**. In this case the neuron has one axon and two dendrites and we shall refer to it as having **order** 3. (The **order** has deep mathematical significance as well because it is equal to the periodicity of the orbit to which points inside the lobe are attracted.)

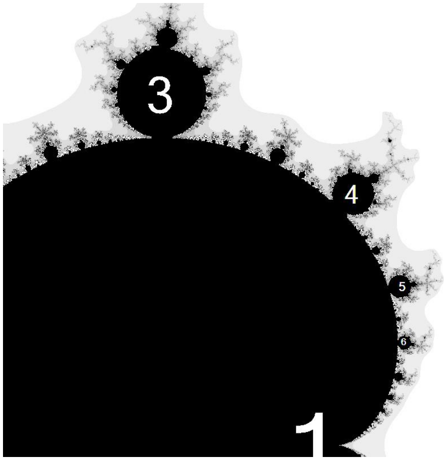
Elephant deep in Elephant Valley
Lobe 21



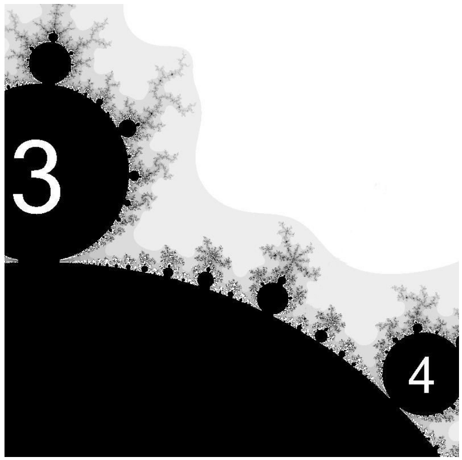
Labelling secondary lobes

Now lets have a look at the upper right hand side of the main lobe.

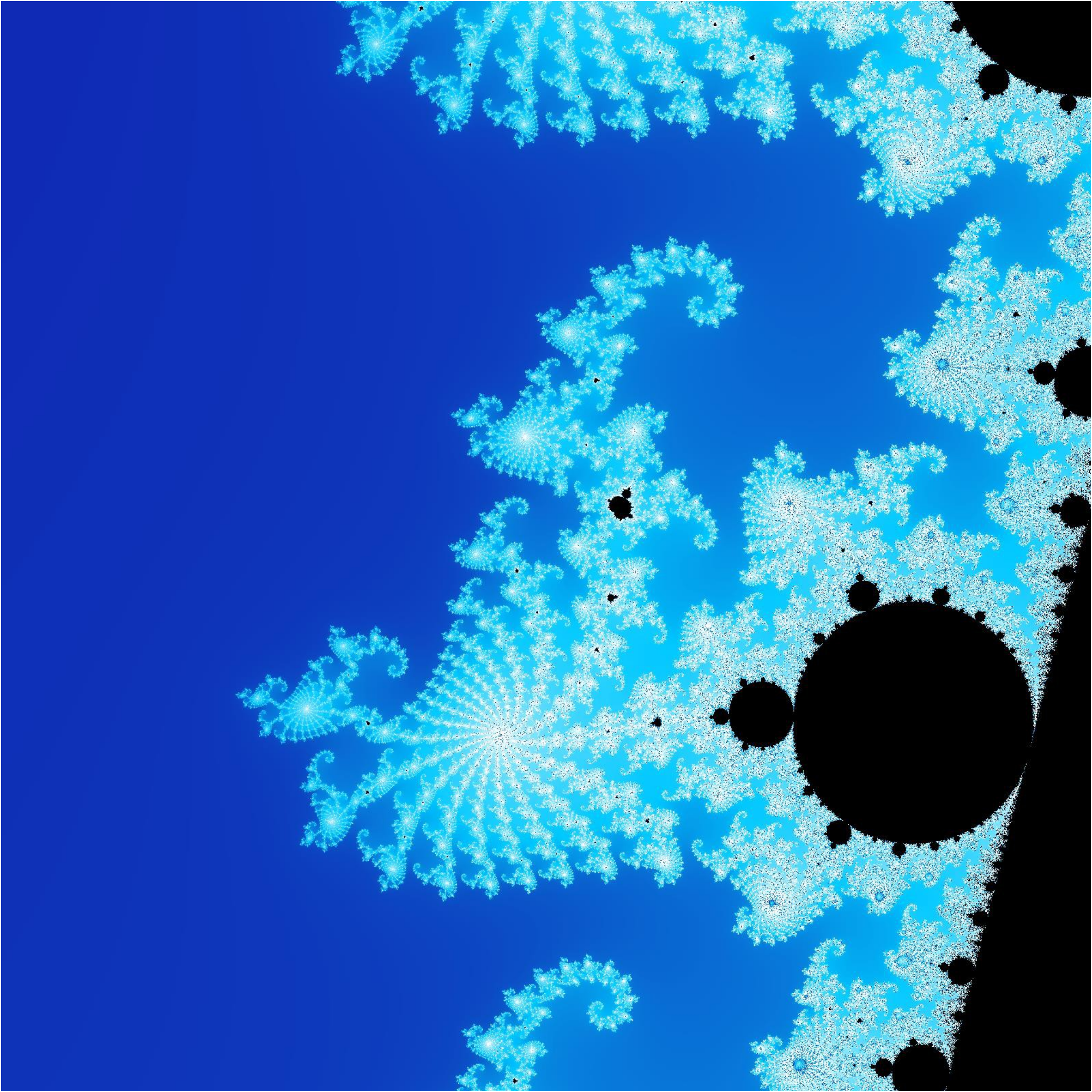
If you look closely at the synapses attached to the lobes which I have labelled 4, 5, 6 etc. you will see that they are of order 4, 5 and 6 etc. In fact there is an infinite series of lobes of diminishing size all the way down to the cusp of the main lobe. We shall call these lobes the **primary** lobes and the whole sequence the **primary sequence of lobe 3**. The problem is – we have now used up all the integers – but there are still a huge number of lobes unlabelled. Look at the region between lobes 3 and 4



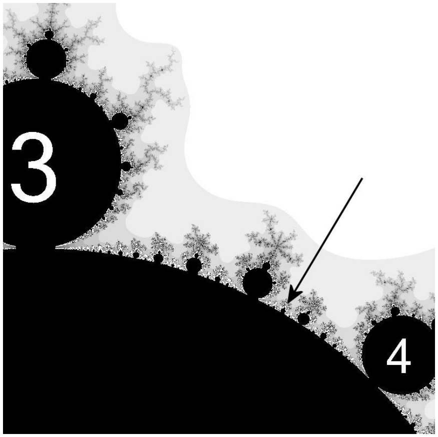
Approximately halfway between lobes 3 and 4 there is the next largest lobe. What shall we call it? You might be tempted to call it lobe 3.5 being halfway between lobes 3 and 4 but there is a good reason why we should not do this. We have seen that lobes 3 and 4 are part of the *primary* sequence of lobes travelling clockwise round the main cardioid. Now the logical way to get to the lobe we are interested in is to go to lobe 4 first, then *change direction* to get to the lobe. This is because the lobe is the second lobe in a *secondary* sequence of lobes starting at lobe 4 and going anticlockwise towards lobe 3. At first sight it might appear that the the lobe is the start of two sequences, one on each side, of equal status. This is not the case. The anticlockwise sequence consisting of lobes is a *secondary* sequence. We shall call 4, 4›3, 4›3›3, 4›3›3›3 … {3} etc. (or more succinctly 4›3^n) the › sign meaning 'move towards'. The clockwise sequence which we shall label 4›3, 4›3›4, 4›3›4›4 … {4} is a *tertiary* sequence and lobe 3 is definitely not part of it. (This sequence can be written 4›3›4^n)



Sea Horse Valley
Lobe 3›2^8



I have marked one of the even smaller lobes with an arrow. What is its label? Well, to get there we should go first to lobe 4, then to 4›3, then another step towards lobe 4 will get us to 4›3›4, but what about the next step? You might be tempted to just add another 3 to the end but this is not correct for reasons which will soon become clear. The next step is taking us not towards lobe 3 but to lobe 4›3. The correct label is therefore 4›3›4›(4›3) Note that the brackets are essential.

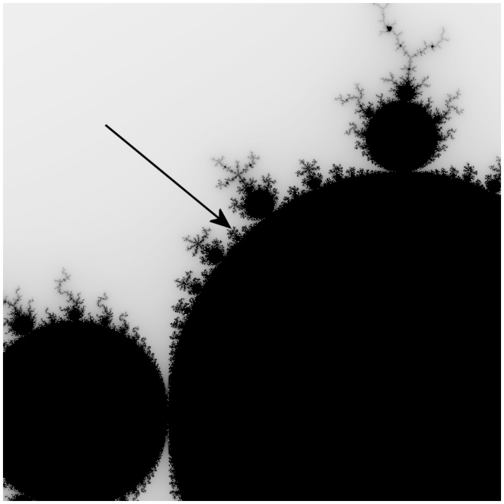


Now if you use your Mandelbrot program to zoom in on the principal synapses of these lobes you will discover that the order of lobe 4›3 is 7 and the orders of the members of the sequence 4›3^n are 4, 7, 10, 13 … (remember, the order of a lobe is the total number of branches round the principal synapse). It is immediately obvious that the order of a lobe is the sum of all the numbers in its Linton label. (e.g. the order of lobe 4›3^4 or 4.3.3.3.3 is 16.)

What about the order of 4›3›4›(4›3)? If you count the branches carefully you will find that there are exactly 18 which is 4+3+4+4+3.

Now lets turn our attention to the upper left hand side of the main lobe. The primary sequence on this side stretches from lobe 3 to lobe 2. Its members are therefore labelled 3›2, 3›2›2, 3›2›2›2 ... {2} and its general form is 3›2^n.

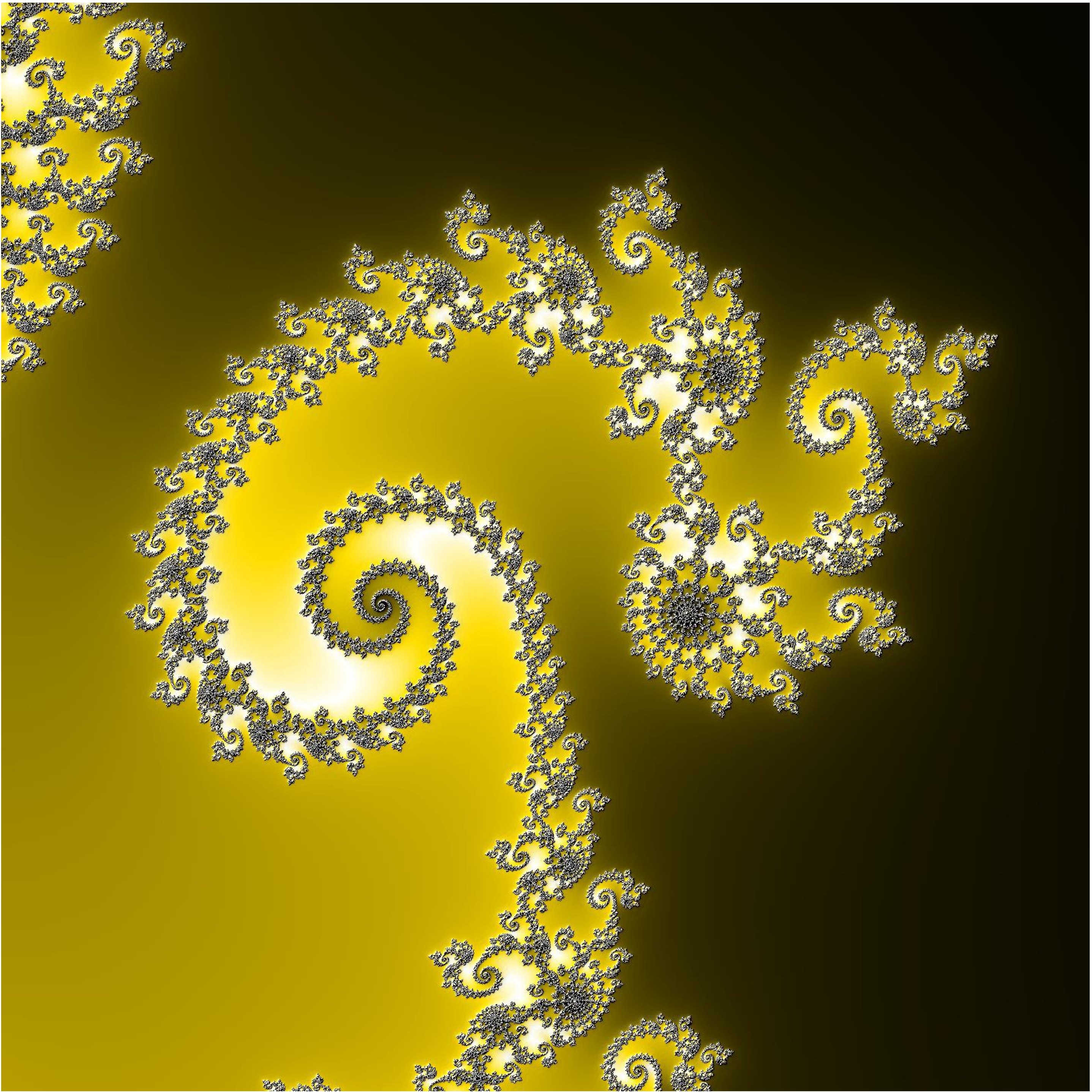
Can you work out the correct label for the arrowed lobe and predict its order? (The answer is below)[1]



Candlestick Bracket in Double Spiral Valley

Lobe 2:11

1 3›$2^{2›}$(3›2) order 12



Labelling the lobes on the bottom of the cardioid is a bit counter-intuitive. The lobe opposite lobe 3 is not, as you might expect lobe -3 or anything like that. It is, in fact, lobe 2›1. The reason for this is that in order to get there starting from the cusp of lobe 1, you have to go one step anticlockwise to lobe 2 and the one step clockwise to get back to the lobe we are interested in.

To summarise what we have discovered so far, every lobe attached to the main cardioid can be given a unique label (the 'Linton label') which effectively describes a path of alternating clockwise and anti-clockwise sequences which have to be taken to reach the lobe. Every lobe is part of either a clockwise or an anticlockwise sequence; the clockwise sequences being primary, tertiary etc. and the anticlockwise sequences secondary, quaternary etc. and every lobe is the start of a sequence going in the opposite direction which terminates at (strictly just before) the previous member of the original sequence.

We are now in a position to state:

The first Mandelbrot theorem

- *The order of any lobe in a sequence is the sum of the orders of the previous lobe in the sequence plus the order of the lobe immediately beyond the limit of the sequence.*

This can be stated in a slightly different way as follows:

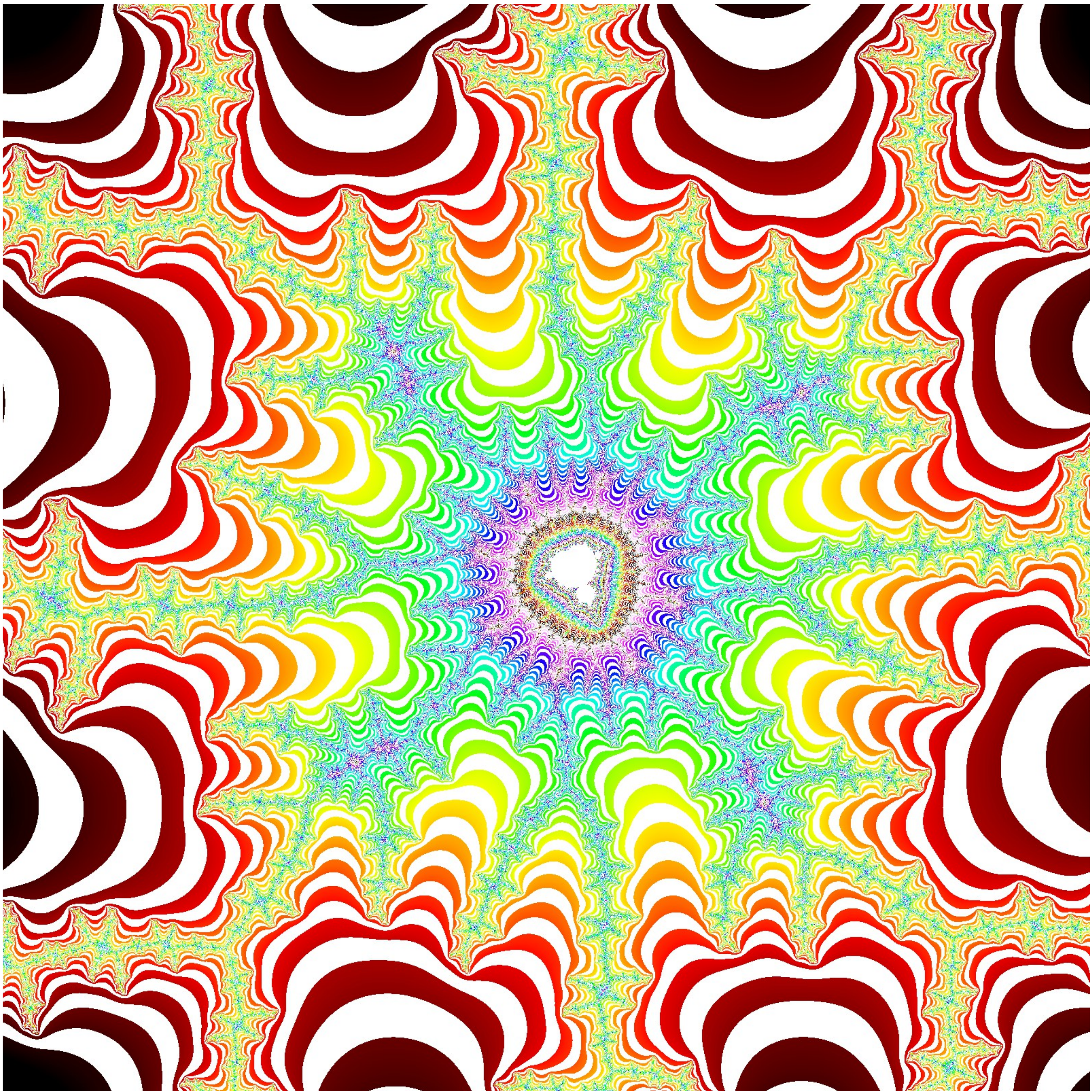
- *The n^{th} lobe of a sequence which starts with a lobe of order A and terminates at a lobe with order B has order $A + nB$.*

If we define the order of the main cardioid as 1 and the largest circular lobe as 2, then we can use this theorem to predict the order of any lobe on the cardioid, however small.

(It is worth noting that even lobe 2 can be regarded as part of the primary sequence which starts at lobe 1 and terminates at lobe 1.)

Banded minibrot

Lobe 3 dendrite

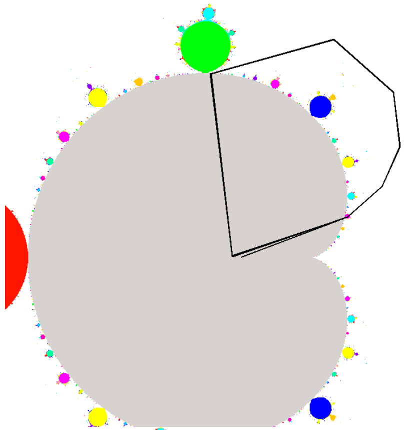


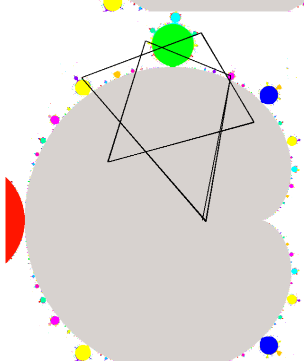
Periodicity and order

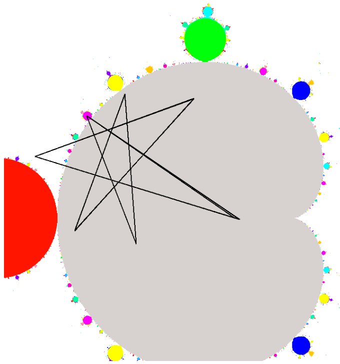
The Mandelbrot Map is generating by iterating the function **z** => $\mathbf{z}^2 + \mathbf{C}$ starting at the point **z** = 0 for all **C** in the complex plane. When **C** is *inside* the main brot (and all the other minibrots too) the point eventually settles down to a steady series of repeating values. When **C** is inside the main cardioid (lobe 1) it converges on a single point. When **C** is inside lobe 2 it jumps between 2 constant values. In general, when **C** is inside a lobe of periodicity (or order) N, the point jumps between N values.

We have already seen that different lobes can have the same order – for example lobes 7, 4›3 and 3›2^2 all have order 7 and it interesting to trace out the path of **z** as shown in the following diagrams:

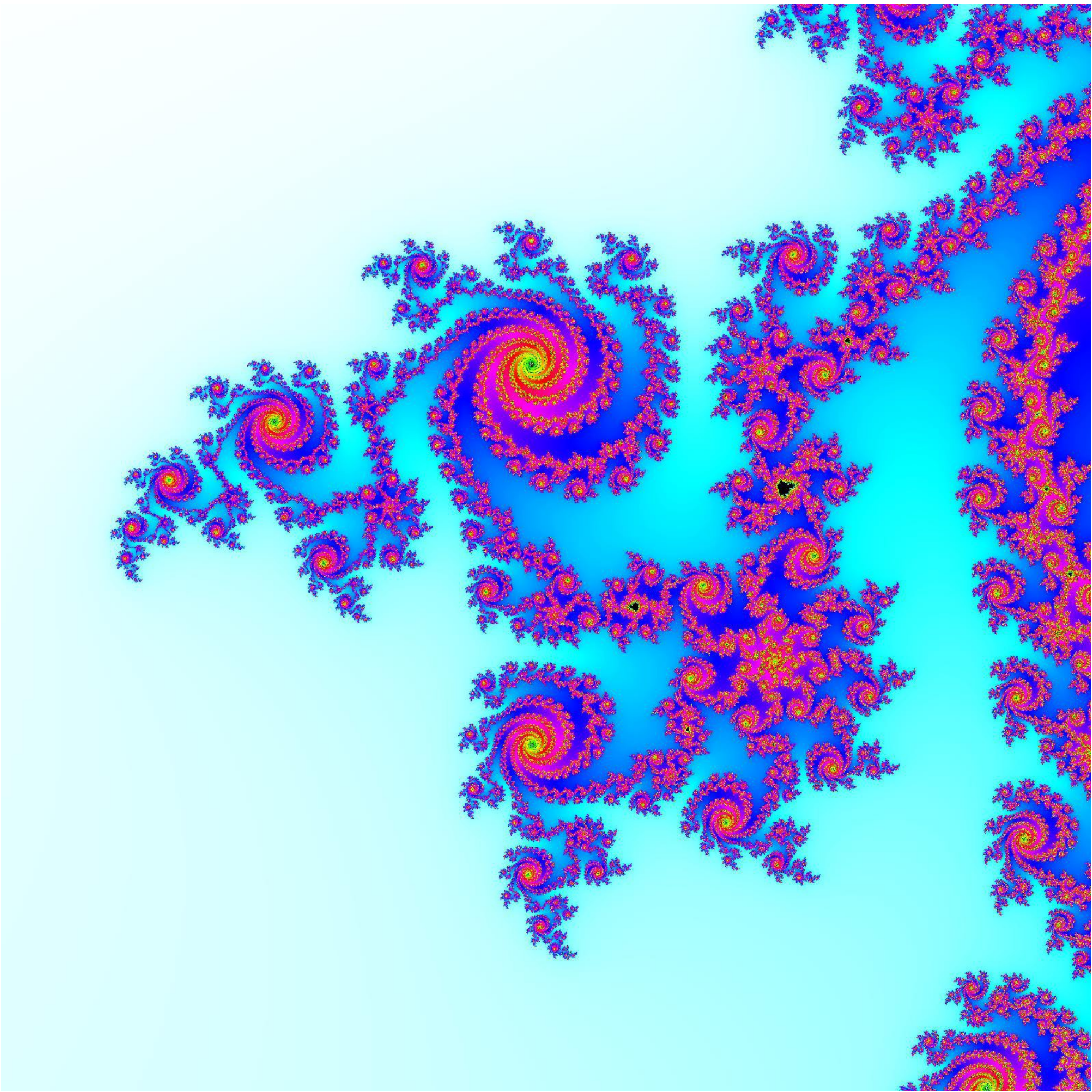
In the first case (lobe 7) the point **z** steps round the sequence one step at a time tracing out an irregular heptagon. In the second case (lobe 4.3) it steps 2 at a time while in the third it steps 3 at a time making a star shape. Now if you look at the star patterns generated by any sequence of lobes between lobes 3 and 4 you will always find that the step size increases by 1 every lobe down the sequence. Take, for example. The sequence 4›3›4^n. Lobe 4, being a primary lobe, has a step size of 1. Lobe 4›3 is one step down the sequence 4›3^n so its step size is 2. Lobe 4›3›4 will have a step size of 3; lobe 4›3›4›4 has a step size of 4 etc.







Butterfly wing
Lobe 5:7



So perhaps the step size is just an indication of how many steps you have to take to get to the lobe in question. Not so. Have a look at lobe 4›3›4›(4›3). You will recall that this lobe has order 18. Now since it takes 4 steps to reach, you might think that its step size will be 4. In fact it is 5. The reason is as follows. When you step from 4 to 4›3 you are stepping towards a lobe (3) whose step is 1. So the step size increases by 1. Likewise when going from 4›3 to 4›3›4 you are stepping towards another lobe of step size 1 so the step size increases by 1 again. But when you take the final step from 4›3›4 to 4›3›4›(4›3) you are stepping towards a lobe whose step size is 2 so the step size increments by 2. In fact, it is just the same as the rule for orders, namely:

The second Mandelbrot theorem

- *The step size of any lobe in a sequence is the sum of the step sizes of the previous lobe in the sequence plus the step size of the lobe immediately beyond the limit of the sequence.*

Now it may occur to you straight away that if the rule for step sizes is exactly the same as the rule for orders, then the step size of any lobe ought to be the same as its order. This is clearly not the case. But why?

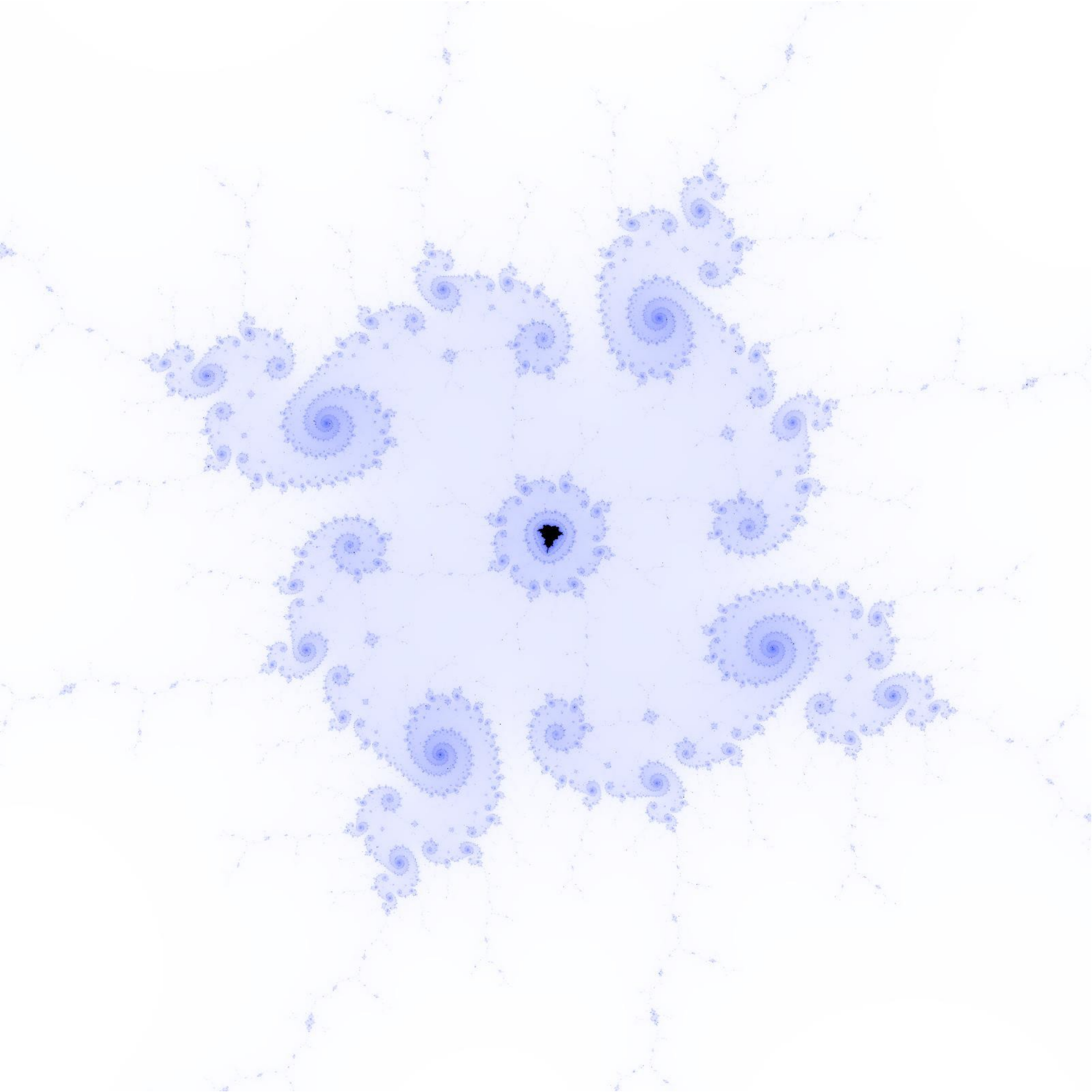
The answer lies in the fact that lobe 1 is both the start and the terminus of the primary sequence of lobes 1, 2, 3, 4, … {1}. It is clear that the order of lobe 1 is equal to 1 – but what is its step size? Since an order 1 lobe doesn't step anywhere, we can assign the step size how we like. Let us assign it a step size of 1 for the purposes of starting the sequence off, but assign it a step size of 0 when it comes to adding extra steps in the sequence. Immediately all the members of the primary sequence will have a step size of 1. Everything else follows.

Calculating the step size from the Linton label is easy. Simply expand all the exponents and then count the number of numbers! e.g. the step size of lobe 3›2^4 will be [3›2›2›2›2] = 5. The step size of 4›3›4›$(4›3)^2$ will be [4›3›4›(4›3)›(4›3)] = 7

There is one interesting consequence of all this: there is only one lobe whose order is A and step size B (A and B being co-prime). For example, while there are are three different lobes of order 7 (with step sizes 1, 2 and 3) you will only find one lobe of order 6, because there is only one way of stepping round 6 points, 6 being divisible by both 2 and 3. On the other hand, there are 2 lobes of order 8 because 8 and 3 are co-prime.

China plate

Tendril of lobe 2:12



We can state this in the form of a theorem as follows:

The Third Mandelbrot Theorem

- *Every lobe is associated with a unique fraction A/B called the rotation number where A is the order of the lobe and B is the step size.*
- *Conversely, for every rational fraction A/B there exists a unique lobe.*

It is easy to calculate the fraction from the Linton label but doing the reverse is a bit more difficult. Lets take an example. The fraction 7/23 defines a lobe with step size 7 and or5der 23. Its Linton label therefore consists of 7 numbers which add up to 23. Now the first thing to notice is that every lobe must lie between two primary lobes with consecutive numbers which in this case must be 3 and 4 (since 7/23 lies between 1⅓ and ¼). The 7 numbers must therefore be 4, 4, 3, 3, 3, 3, and 3. Now 4›3›3 and 4›3›3›3 are two consecutive members of the sequence 4›3^n so the label we are searching for is 4›3›3›(4›3›3›3). Can you calculate the label for the lobe 7/24[2]?

Now I have already suggested that the combination of step size and order defines a unique lobe and hence a unique Linton label but it is not at all clear to me how to calculate the Linton label of, say, lobe 17/61. I am, however, convinced that a) there is an answer and b) that it is unique. (I do know the answer because I invented it and worked out the fraction! You will find it below[3].)

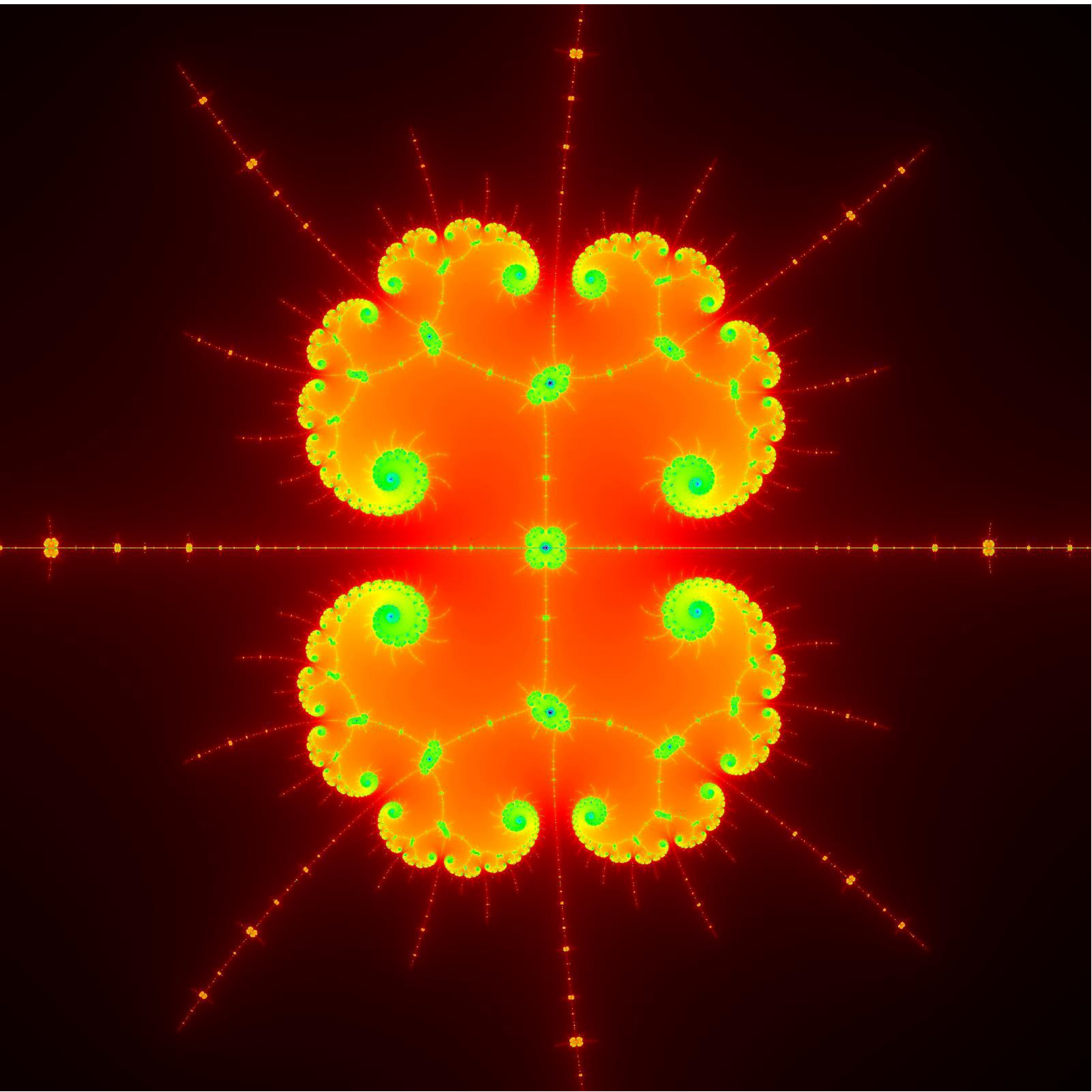
Interestingly, the rules apply equally well to the lobes on the lower half of the cardioid. I have already pointed out that the lobe opposite lobe 3 should be called lobe 2›1. The periodicity of this lobe is therefore 2+1 = 3 and the step size is 2 (because there are 2 numbers in the label). Its rotation number is therefore 2/3. The rotation numbers of all the lobes on the lower half of the cardioid lie between ½ and 1.

2 The lobe 7/24 has the label 4›3$^{2›}$(4›3)^{2}

3 The answer is 4›3›4›(4›3)$^{2›}$(4.3›4›(4›3))^{2}

Nuclear explosion

Axonal minibrot

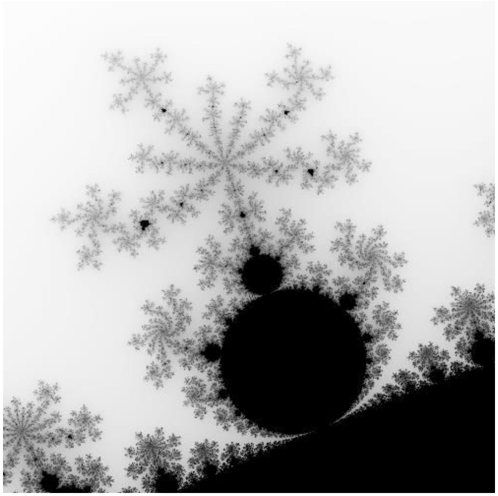


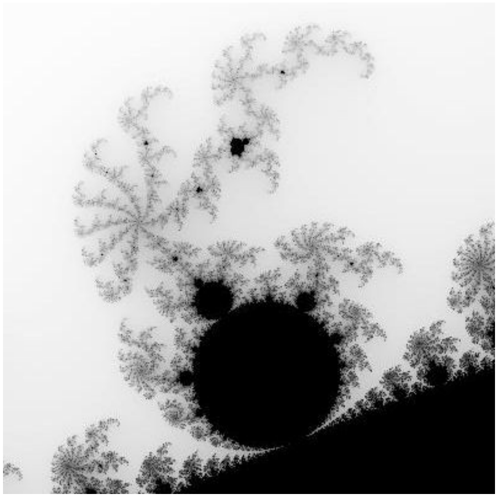
Synapses and step size

The order of a lobe is obvious as soon as you look at the principal synapse. Is there any way we can also determine the step size of the lobe and hence its fraction? Look at the two order 8 lobes shown here:

One of them is lobe 8 with associated fraction 1/8 while the other is lobe 3›2›3 with associated fraction 3/8. But which is which?

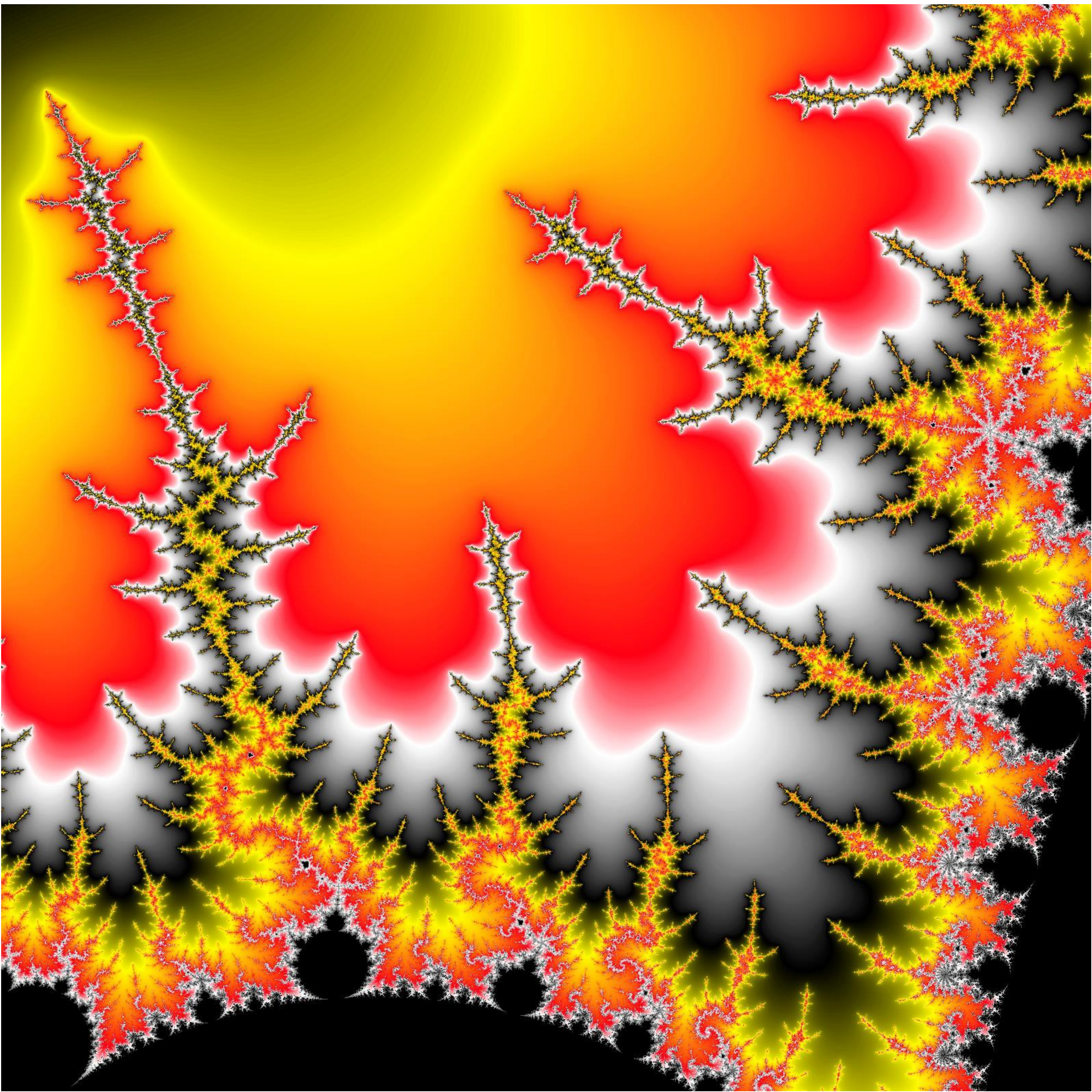
The way to tell is by looking at the way the different dendrites are organised. In the lower image, the dendrites increase in length steadily in a clockwise direction but in the upper one the dendrites seem to be randomly organised. There does not seem to be an easy way, however, to determine the step size of a lobe just by looking at its principal synapse.





Forest fire

Valley between lobes 1 and 2

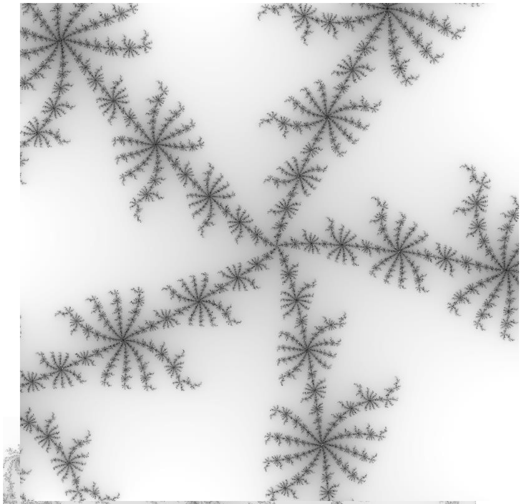


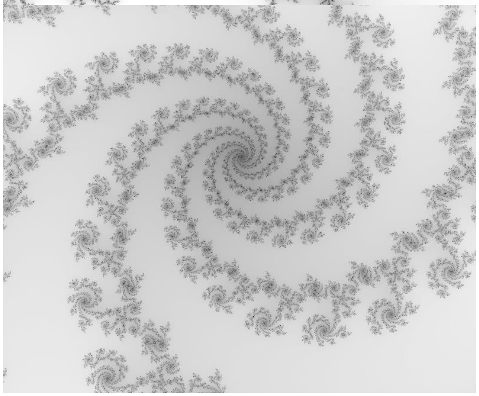
Synapses

What, exactly, are synapses? I have stated above that when **C** is inside any of the main lobes (or any of the surrounding minibrots), the starting point **z** = (0, 0) settles down to a periodic orbit whose periodicity is the order of the lobe (or minibrot). Now it is easily shown that for every value of **C** in the plane, there are an infinite number points which map onto itself after a number of iterations. Some of these points are attractors. For example, if **C** is inside lobe 1, **z** quickly homes in on the point which is one of the solutions to the equation $\mathbf{z}^2 + C = \mathbf{z}$. If, on the other hand, **C** is anywhere outside the set, **z** usually never finds a periodic point and shoots off to infinity.

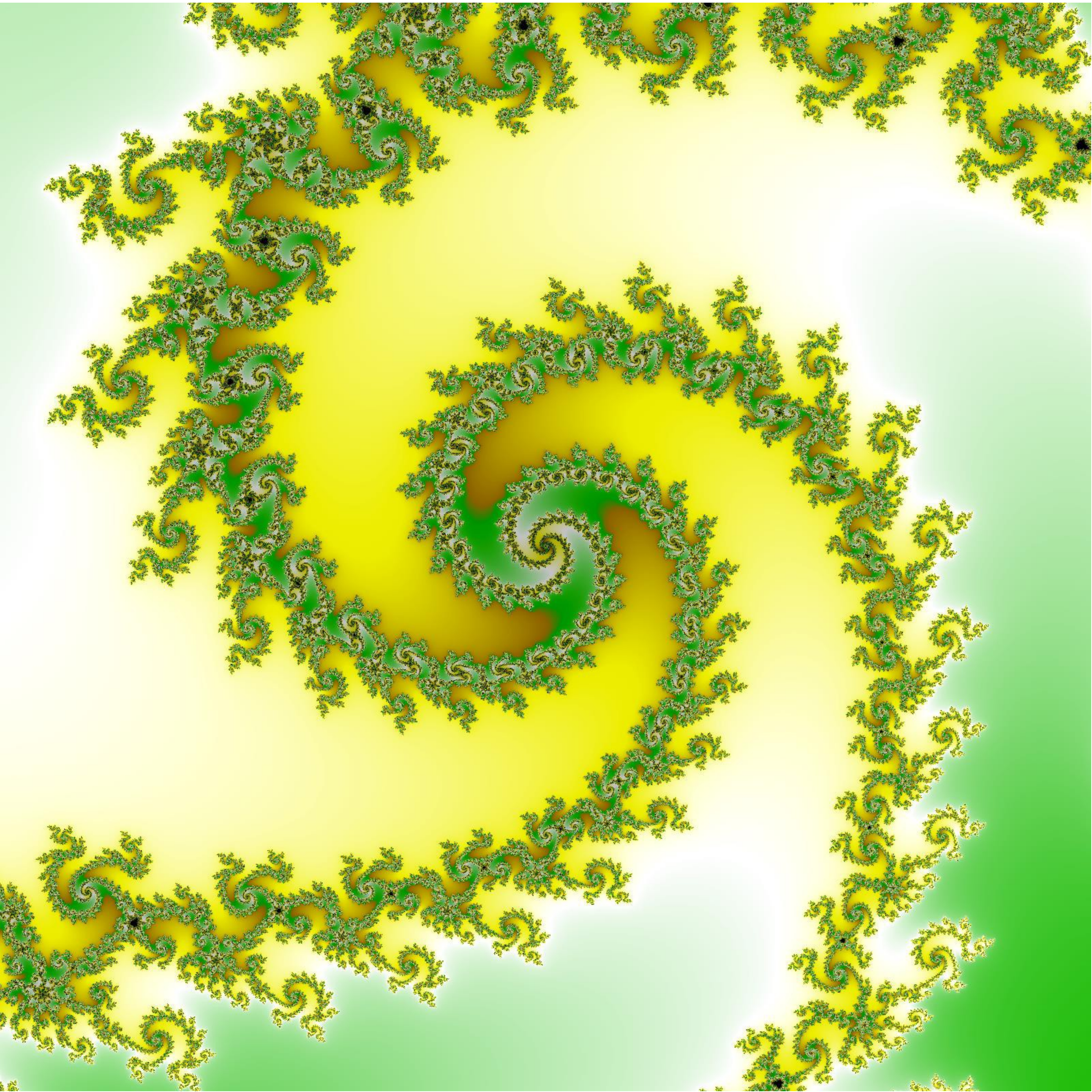
But just occasionally, after wandering around for a bit, **z** may happen to hit precisely on a periodic point. These points are called Misiurewicz points and these are the synapses. If this happens to be a synapse of order 5, then after 5 more iterations **z** will find itself back in exactly the same place and this will happen over and over again. On the other hand, if **C** is even a tiny bit off the true value, **z** rapidly spirals away from the periodic point off to infinity. The point is a repeller. I shall have more to say about Misiurewicz points and synapses when we come to examine the principal axon in more detail.

The images on this page show two synapses of order 5. Both are attached to lobe 5 but the spiral version is found attached to one of the smaller tertiary lobes on lobe 5.





Spiral synapse
in Sea Horse Valley of lobe 3



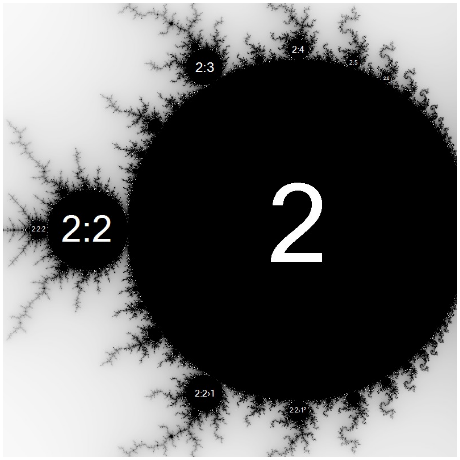
Labelling the lobes on lobes 2 and 3

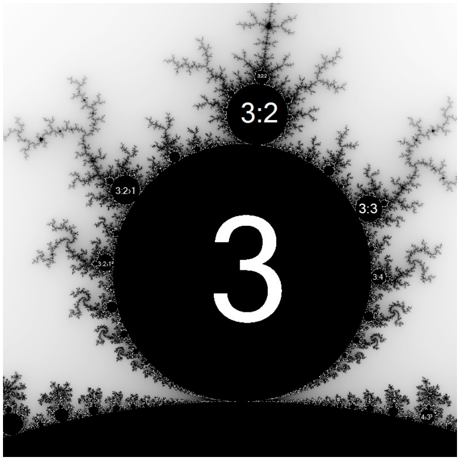
So far we have been talking exclusively about the lobes attached to the main cardioid (i.e. lobe 1). What about the tertiary lobes attached to the secondary lobe 2?

The first thing to notice is that there is exactly the same arrangement of mini-lobes around lobe 2 as there was around lobe 1, the only obvious difference being that the largest lobe on the upper semicircle is as about 10 o'clock rather than 12 o'clock as it was before. (In fact, this is the 'correct' position being ⅓ of the way round the circle. It is the lobes on lobe 1 which are in the 'wrong' places because the cardioid is a sort of distorted circle.) The principal synapse of this lobe still has 3 branches so we can label it as lobe 3 but in order to indicate that it is attached to the primary lobe 2 rather than lobe 1 we shall prefix this with the number 2 and a colon (meaning 'attached to;). The main sequence of lobes on lobe 2 are therefore 2:2, 2:3, 2:4, 2:5 etc. and the arrowed lobe should be labelled 2:3.2. (Note that the periodicity of this lobe is 10, not 5.)

Since the largest mini-lobe on lobe 2 (the one on the X axis) is called 2:2, the main sequence of lobes round this lobe should be 2:2:2, 2:2:3, 2:2:4 etc. Likewise, the main sequence lobes on lobe 3 are called 3:2, 3:3, 3:4 etc. in the clockwise direction and 3:2›1, 3:2›$1^{2,}$ 3:2›1^{3} etc. in the anticlockwise direction.

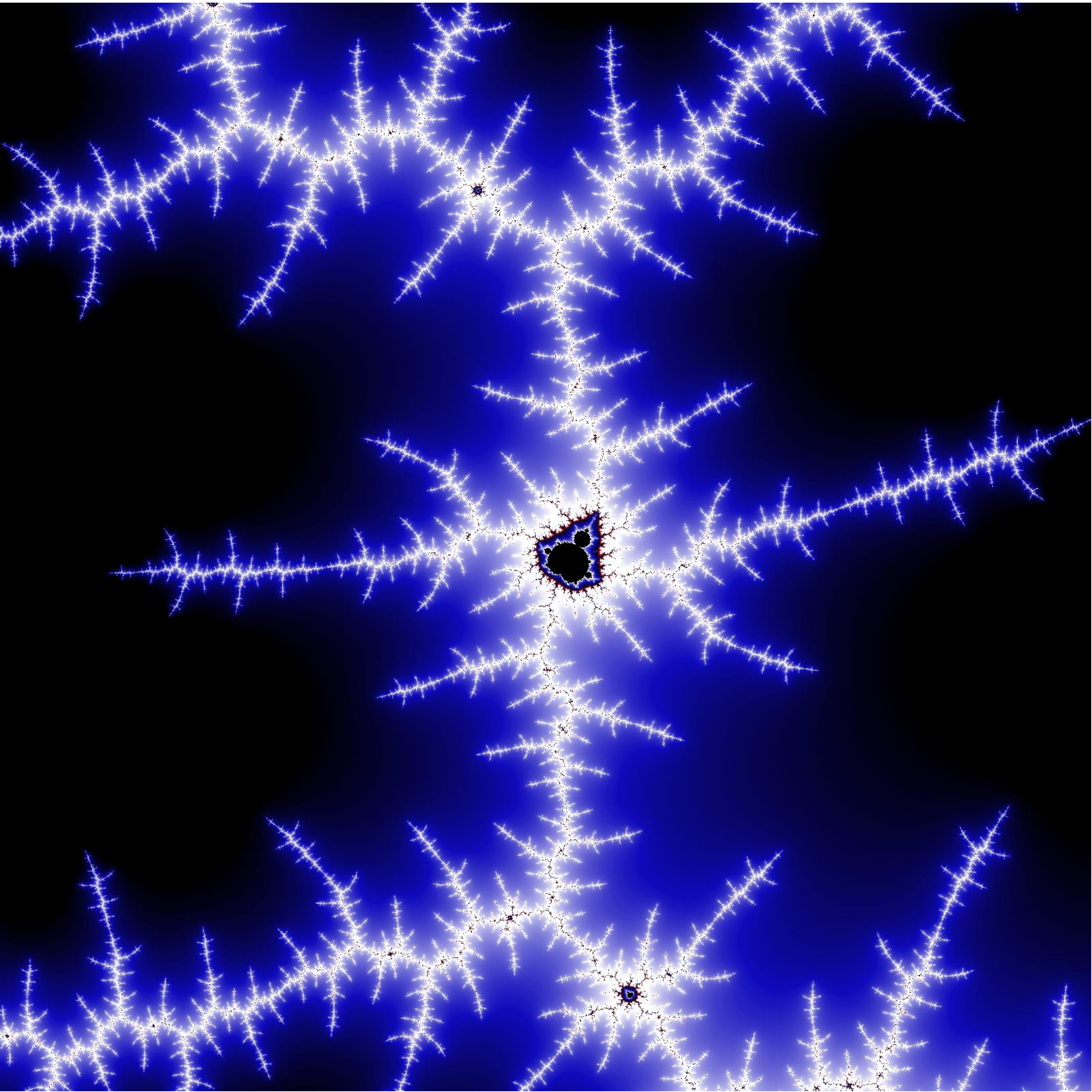
(Strictly speaking, all the mini-lobes should have the prefix 1: as they are all, ultimately, attached to lobe 1; and the arrowed lobe ought to be labelled 1:2:3›2 as it is lobe number 3›2 attached to lobe 2 attached to lobe 1 but we can omit the 1: without loss.)





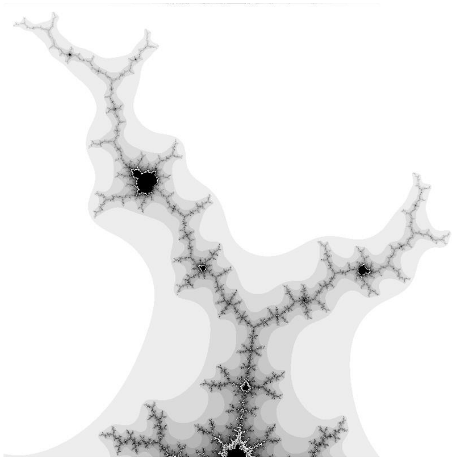
Lightning bolt

Neuron of lobe 2:3



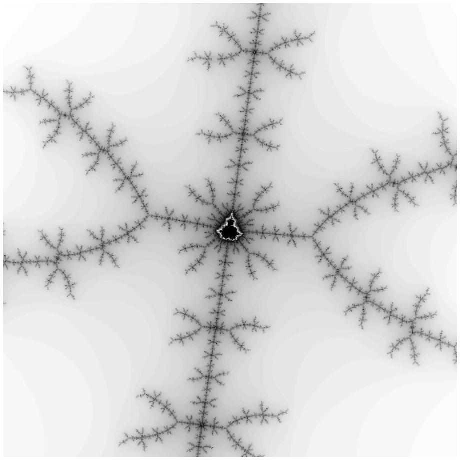
Neurons and their features

Now we know which parts of the map we are talking about, lets start to look at the various different neurons and their features. Here is the principal neuron of lobe 3:

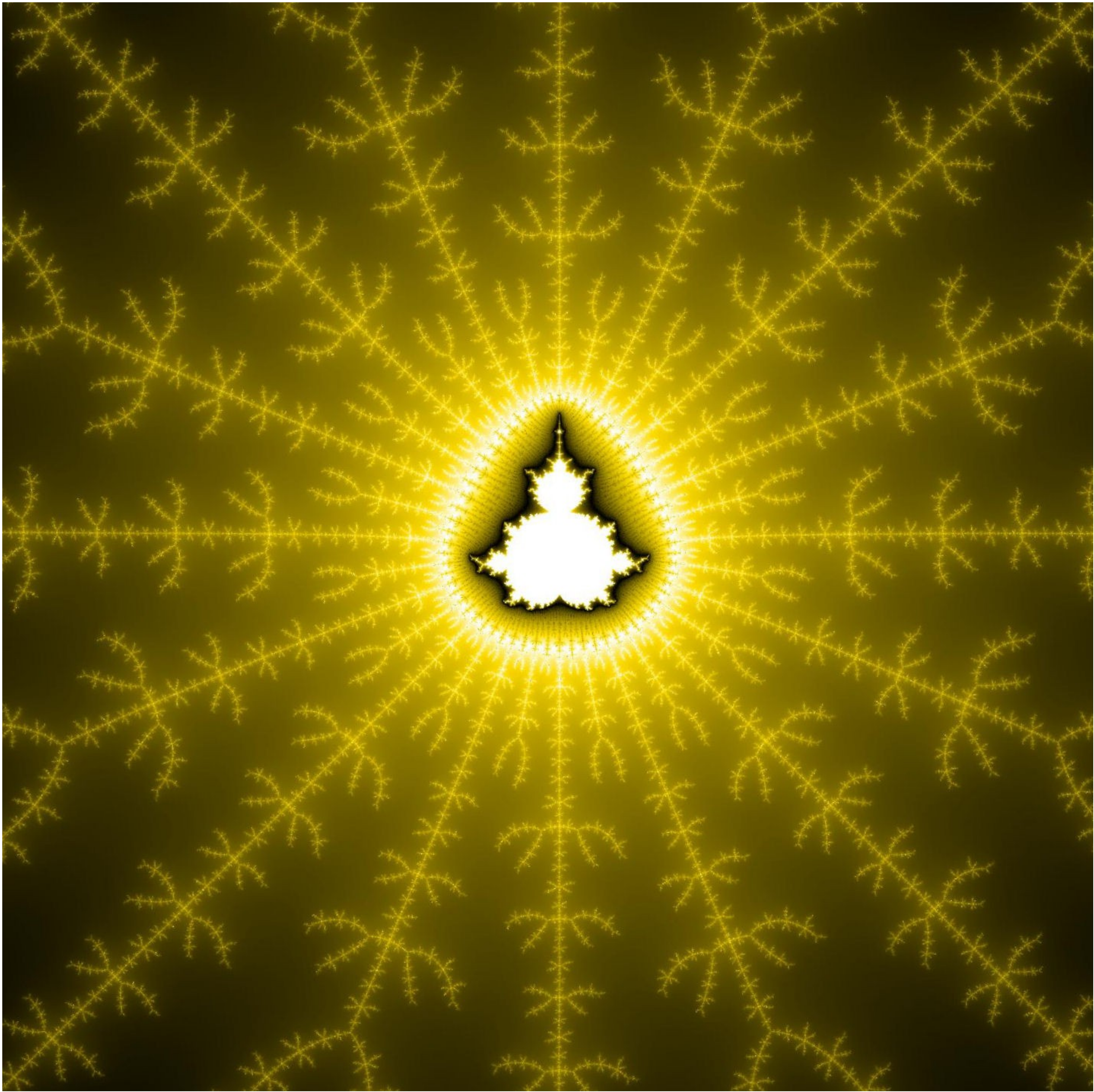


Of the two dendrites it is the first (measured clockwise from the axon which is the longer and leads to the largest minibrot. (Minibrots are small islands of stability with the same general shape as the whole set.) The axon has a minibrot approximately half way between the lobe and the synapse and, if you look closely, you will find an infinite number of other smaller minibrots on either side making a fractal sequence. The whole structure is linked together with a filigree of channels which are believed to have infinite depth. Every single synapse has order 3. You can magnify any synapse as much as you like and you will never reveal anything other than a straightforward junction. If, however, you spot what appears to be a synapse with a higher order – for example, 6 – you will find a minibrot there with two order 3 synapses, one on each side. This is one of the smaller minibrots on the principal axon of lobe 3:

In fact, every minibrot in the map has this basic structure: an axon (or dendrite) running through the middle of the minibrot and two more or less prominent synapses symmetrically on either side. You will notice that four smaller order 3 synapses have appeared in the four quadrants – and eight even smaller ones between them. As you magnify smaller and smaller minibrots, the binary symmetry grows so that eventually you can find minibrots surrounded by 64 or 128 synapses like the one opposite.

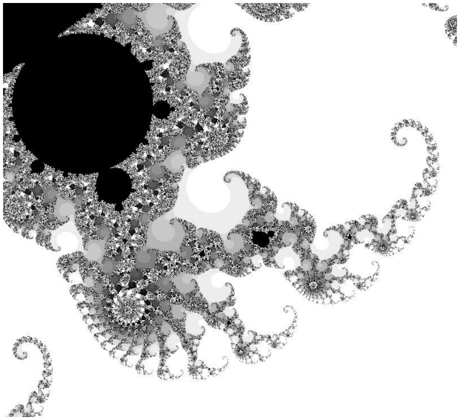


Deep minibrot
on the principal axis of lobe 3



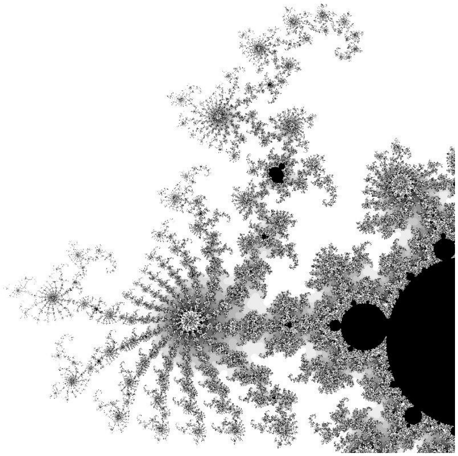
Neurons of high order

Neurons of high order lobes are much more interesting. This is the principal neuron of lobe 15 which is in the region known as 'elephant valley' for obvious reasons (though this elephant happens to be upside down!):



The principal synapse is the spiral structure at the bottom left. The first dendrite (measured clockwise from the axon) is the longest. Halfway along there is a large minibrot. If you trace along its principal axon (the one that emerges from lobe 2 on the minibrot) you will see that it leads to another synapse of order 18 but this time it is dendrite number 17 which is longest and leads via a small minibrot to the next largest synapse. This pattern is repeated continuously on a smaller and smaller scale resulting in a beautiful spiral which seems to disappear down into infinite depths. Every dendrite on every synapse ends in one of these **infinite spirals**.

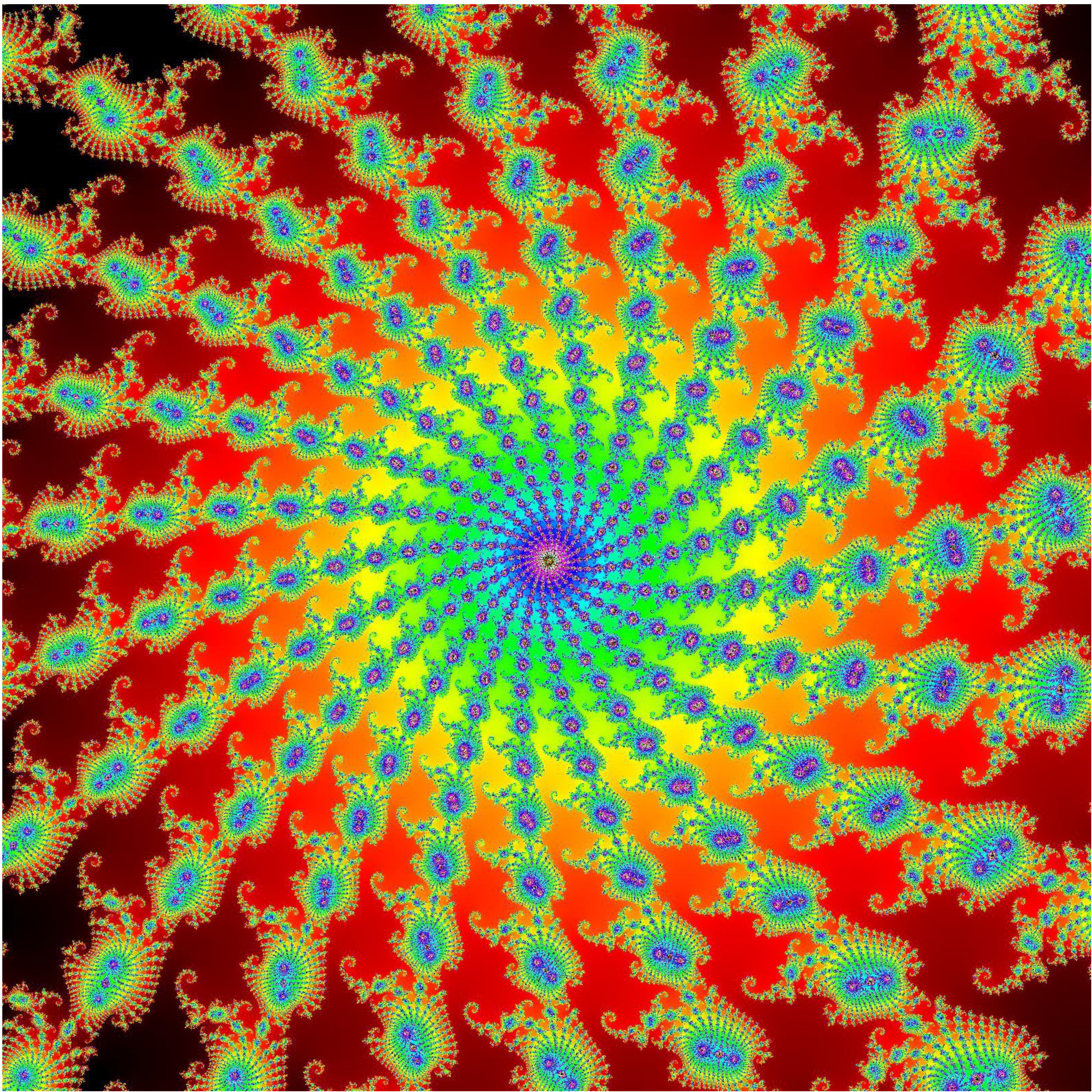
The lower image shows the principal neuron of lobe $3›2^6$ which also has order 15 and step size 7. (You will notice that the 14 dendrites fall into two distinct series of 7. This must be related to the step size of the lobe.) The largest dendrite ends in an infinite spiral forming a curving head which has reminded some people of a sea horse. This wedge-shaped promontory between lobes 1 and 2 is often known as 'sea-horse valley'.



The Peacock Synapse

Lobe $3›2^9$

The minibrots which link the large synapses together are strikingly symmetric and would make a good design for a wallpaper. (Once again, it is worth noting the basic structure of a minibrot, namely and axon or dendrite running through it with two prominent synapses, one on each side.)

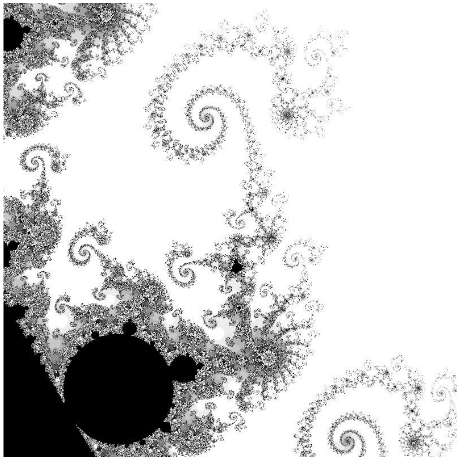


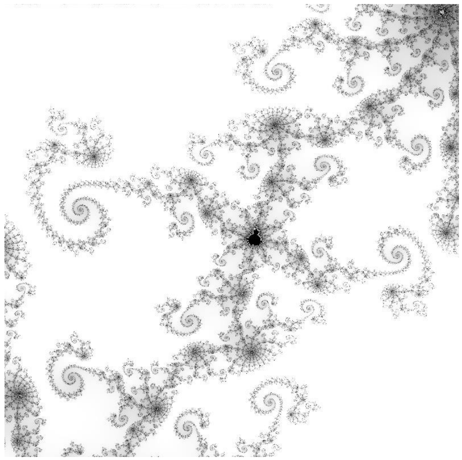
Neurons on the lobes of lobe 2

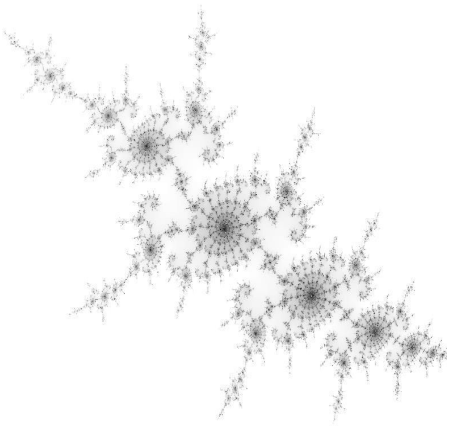
The images below are of the principal neuron of lobe 2:12 which can be found on the opposite side of 'sea horse valley'. (In fact I think these neurons look more like sea horses than the ones on the main lobe.)

You can tell that it is on lobe 2 rather than on the main lobe because the infinite spirals are double: they go in and then come out again! This gives the linking minibrots a particularly pleasing aspect. (The point at the bottom of the spiral is another of those Misiurewicz points and is, in fact, a synapse of order 2.)

Looking on the other side of the lobe (i.e. between lobe 2:2 and lobe 2:3), the neurons frankly look a bit of a mess; but the tips of the dendrites are rather lovely. Here is the tip of dendrite number 8 (the largest one) on lobe 2:3›2^6 and a close up one of the minibrots which link the order-15 synapses:

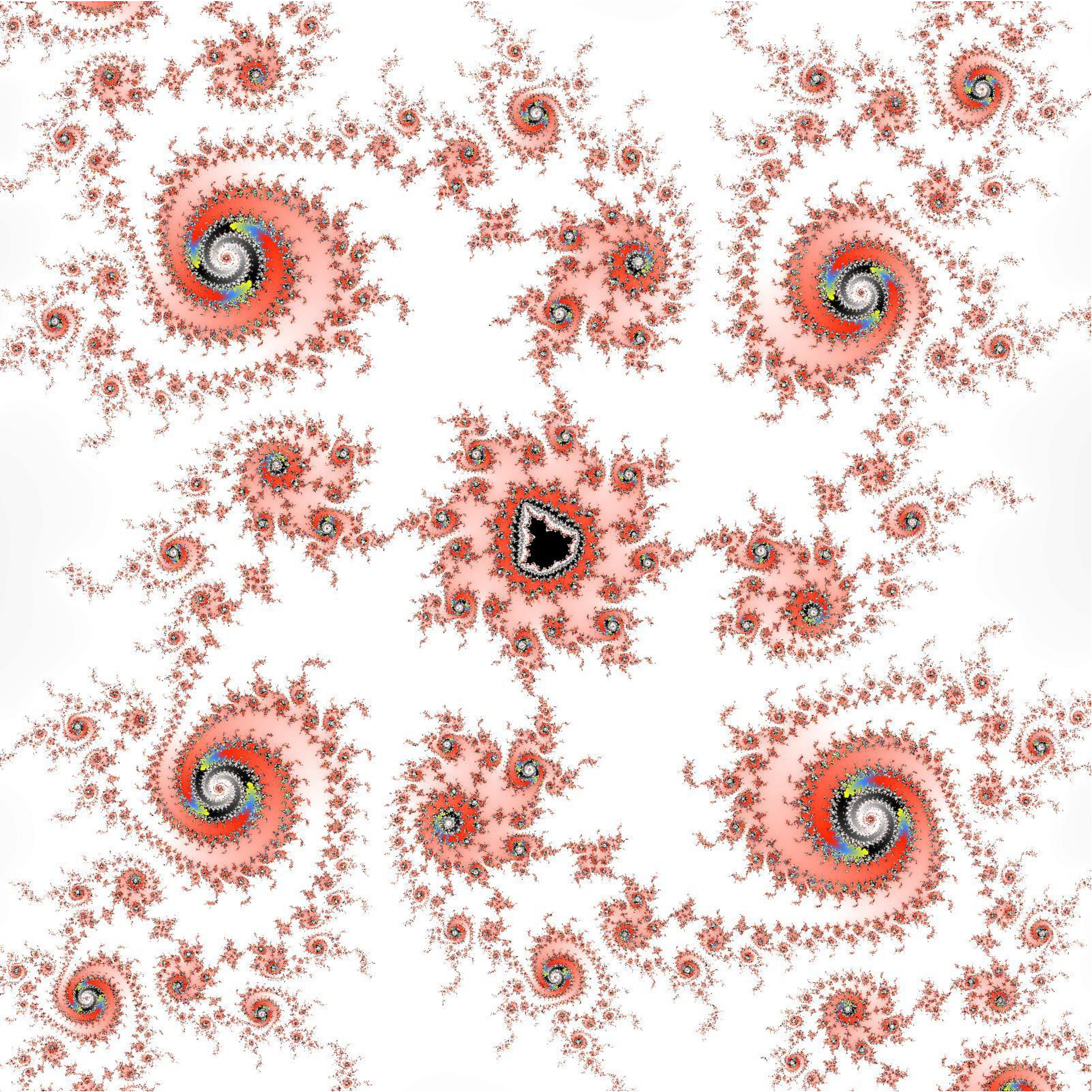






ccc

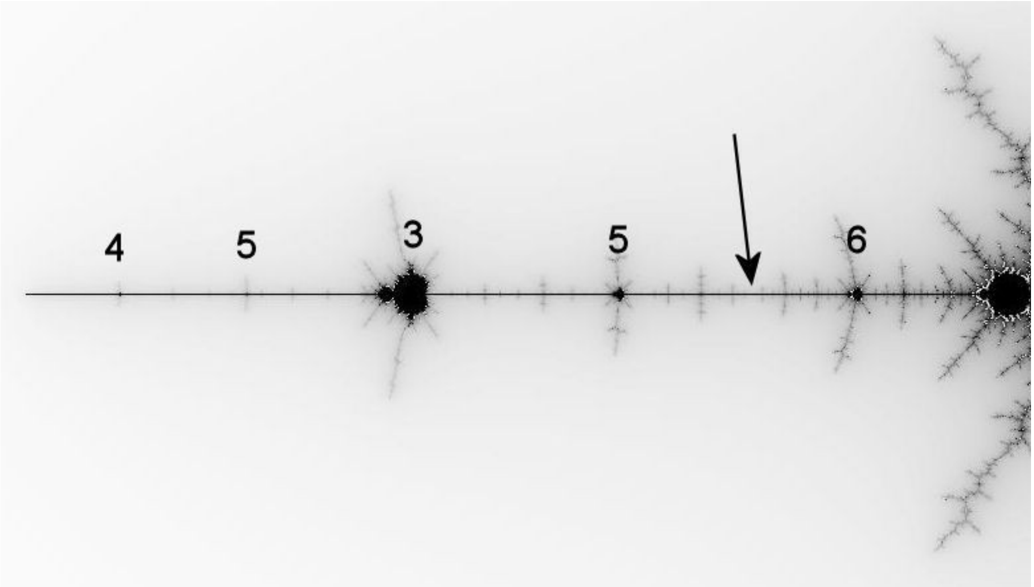
Pretty wallpaper
in one of the sceptres of lobe 2:3:11



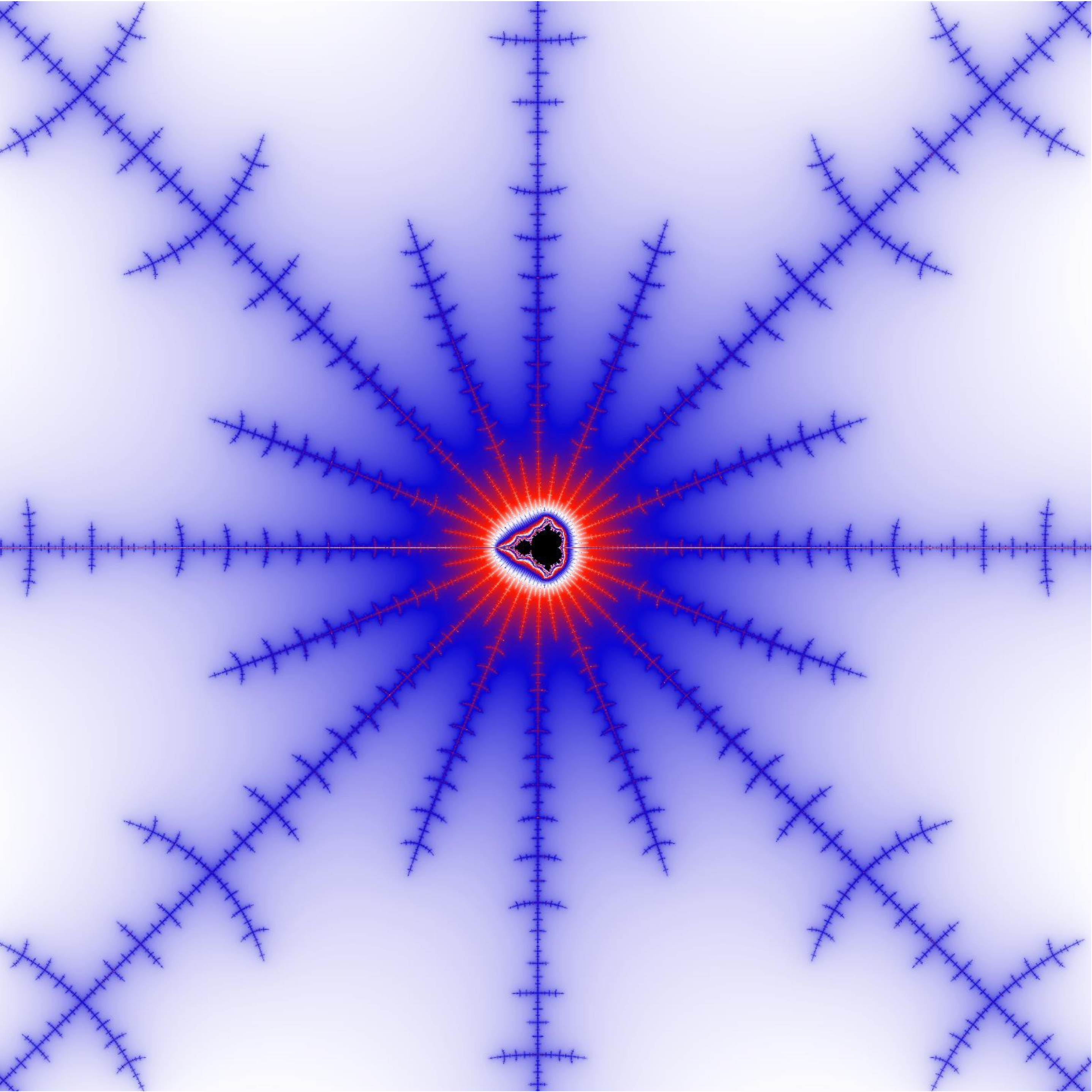
The primary neuron of lobe 2

The primary neuron of lobe 2 – the one along the X axis – is, of course, special. (It is often called the 'Antenna' of the Mandelbrot set but I prefer to refer to it as the **primary axon**.) It has order 2 so all the order 2 synapses lie along a straight line. In fact it is impossible to tell where the synapses are; all you can see is a chain of minibrots of varying sizes. Each of these minibrots is organised in the same way as the main one. The principal synapse is approximately half way between the main lobe and the largest minibrot on the axis and is shown with an arrow in the image below.

It is immediately obvious that the fractal structure of the axon implies that there are an infinite number of minibrots; indeed, there are an infinite number of minibrots between *any* two minibrots. But are they connected together? That is the question. And where are the synapses?

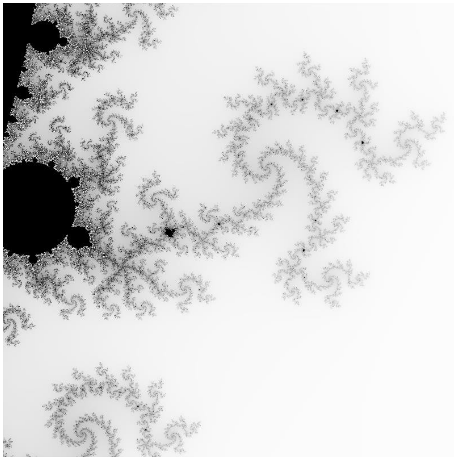


The periodicities of the larger minibrots along the antenna



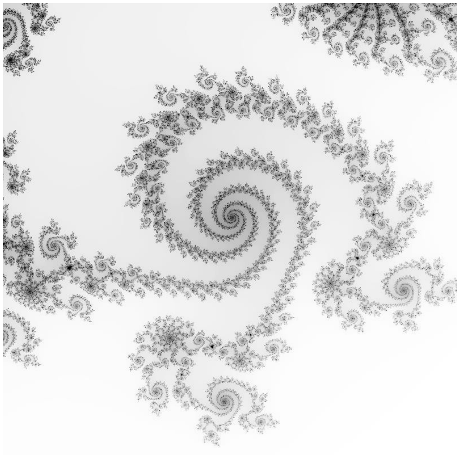
Neurons on the secondary lobes of lobe 3

We now turn out attention back to lobe 3. We know that this lobe has order 3 but what does this mean in terms of its secondary lobes? Here is an image of lobe 3:5:



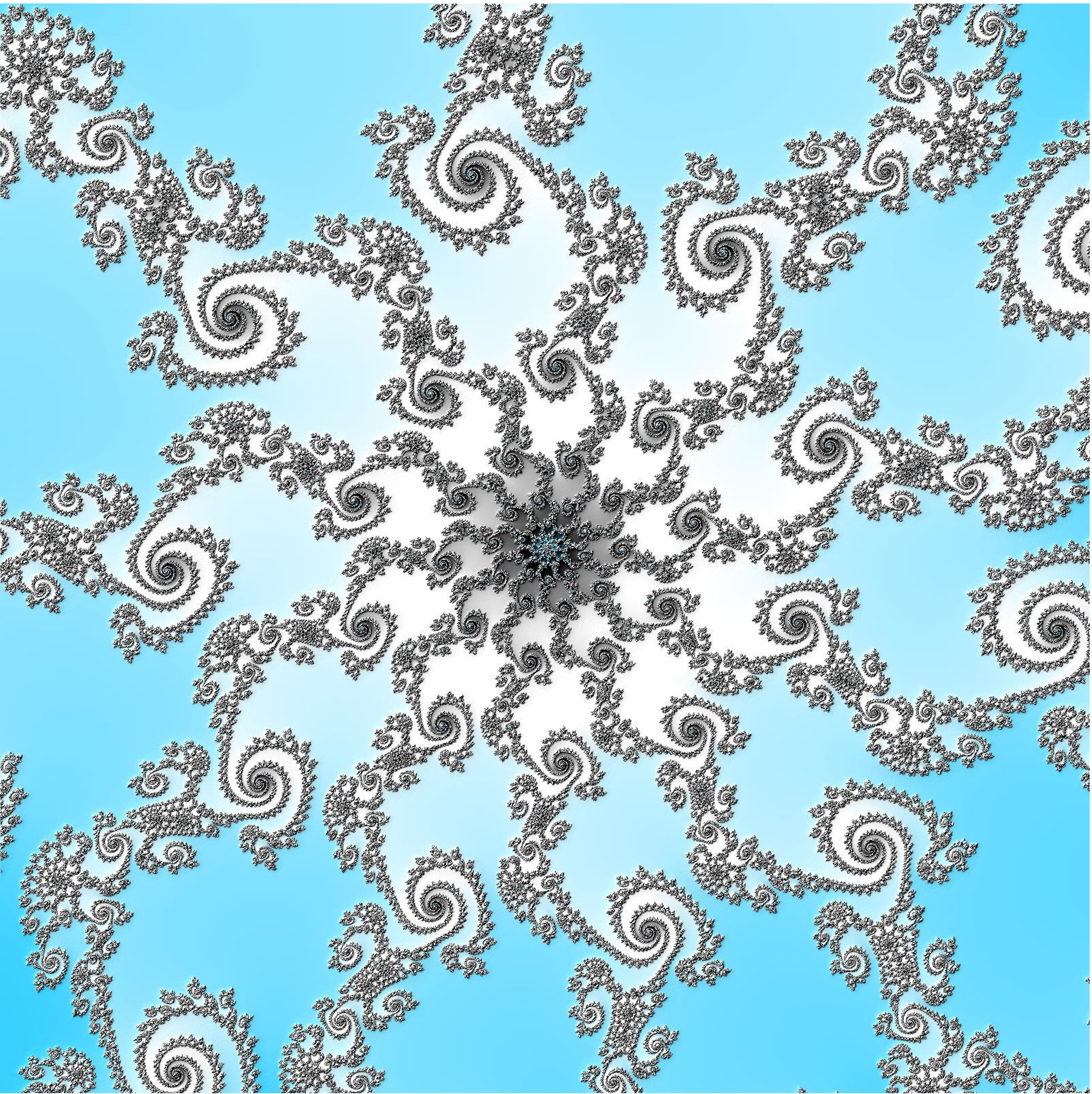
It is immediately obvious that we have synapses both of order 5 and order 3. The **principal synapse** is of order 5 and is attached to the lobe by a short straight axon but we can also see a beautiful triple spiral structure on the end of every dendrite which is, of course, a synapse of order 3. If you look at any of the dendrites in more detail you will find many more 'straight' order 5 synapses and spiral ones of order 3.

When we looked at the the secondary lobes of lobe 2 we saw double spirals. Here is the spiral synapse of lobe 3:10. It is still an order 3 synapse but it is given a lot more twists because of the high order of the lobe to which it is attached.



The lower image shows the principal (order 3) spiral synapse of lobe 3:10 and the coloured image shows a close up of one of its many order 10 synapses. It is worth noting that the order 10 synapses are **radial** rather than spiral and, like all synapses, they are self similar so you won't find a minibrot at the bottom of them. You will, however, find lots of beautiful minibrots in the arms of the synapse.

Wedgwood plate
Lobe 3:10



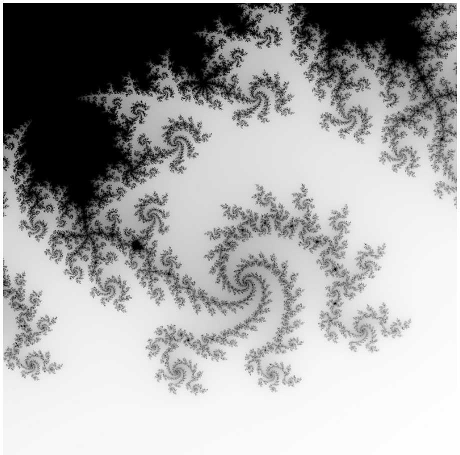
We can now appreciate better why there is always an absolutely straight section between every lobe and the first major synapse. The principal neuron of lobe 3/5 does not actually emerge directly from that lobe because, as you can see from the image below, there are an infinite number of order 2 lobes stacked one after another so the lobe which the axon actually emerges from is 3:5:2:2:2... What this means is that you have to pass through an infinite number of order 2 synapses before you reach the first order 5 synapse, passing a large minibrot on the way. These synapses are always perfectly straight. Then, after several twists and turns and another large minibrot, you come to the order 3 synapse.

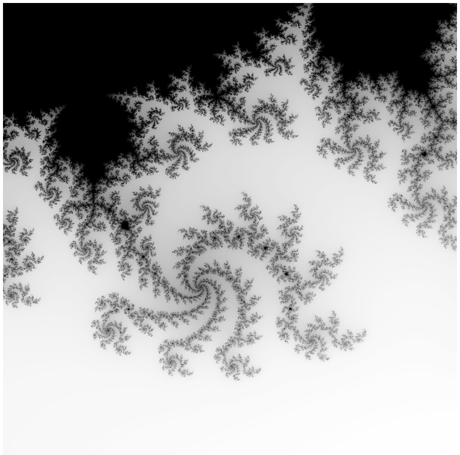
Bearing in mind that the real label for this lobe is 1:3:5:2:2:2... we should expect to come eventually to an order 1 synapse. But what is a synapse of order 1? It is a synapse with only one branch – a terminus! We can even find spiral synapses of order 1. Where would you expect to find them? At the end of an elephant's trunk, of course!

Can you tell which lobes these images are of?

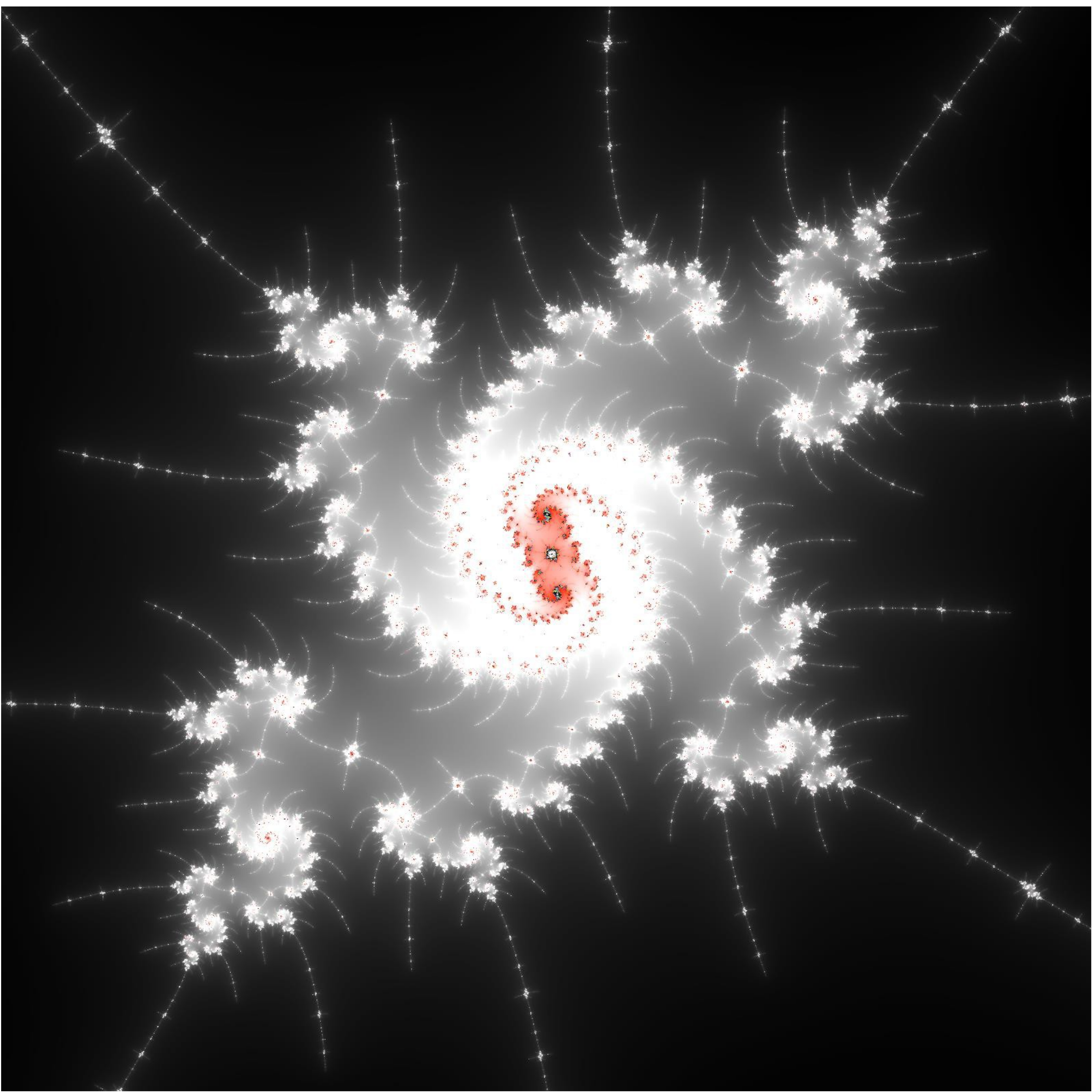
The most obvious feature is the spiral synapse but this is not really the most important feature. That is the first synapse at the end of the axon – the **principal synapse**. This is the order of the lobe. The upper one has order 5 and the lower one has order 4. Knowing the order of a lobe does not tell us uniquely which actual lobe it is because, as we have seen, different lobes can have the same order but I can tell you that these are both principal lobes.

The spiral synapses are of order 4 and 5 respectively so the lobes are 4:5 above and 5:4 below.



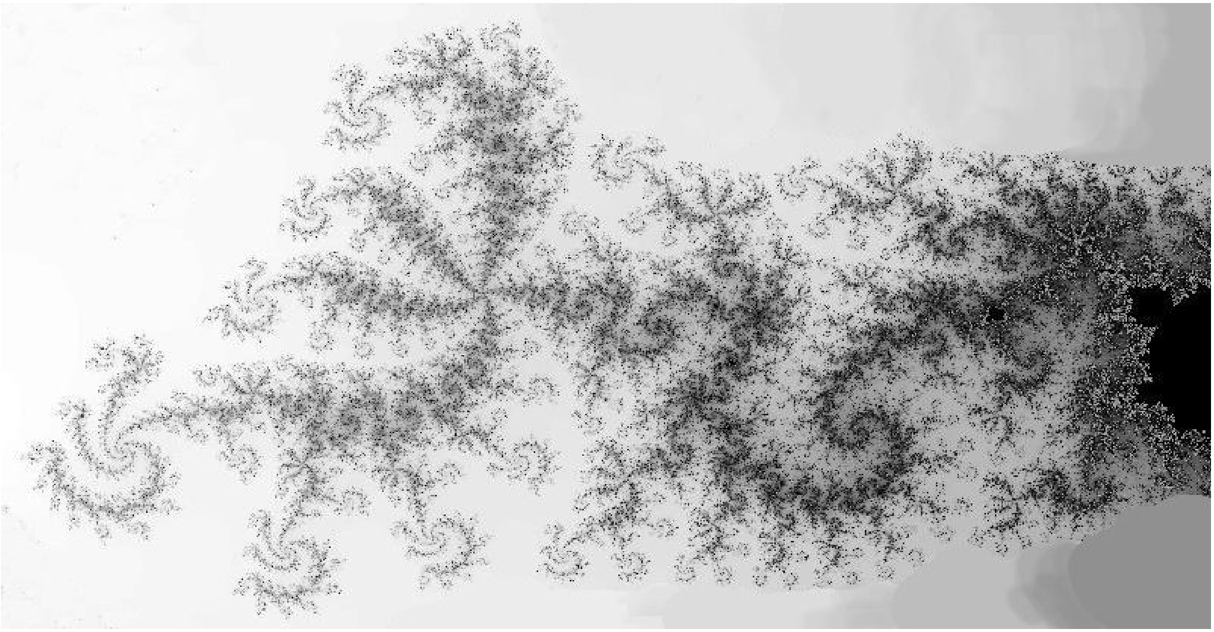


Spiral Galaxy
deep in one of the tendrils of lobe 2:7

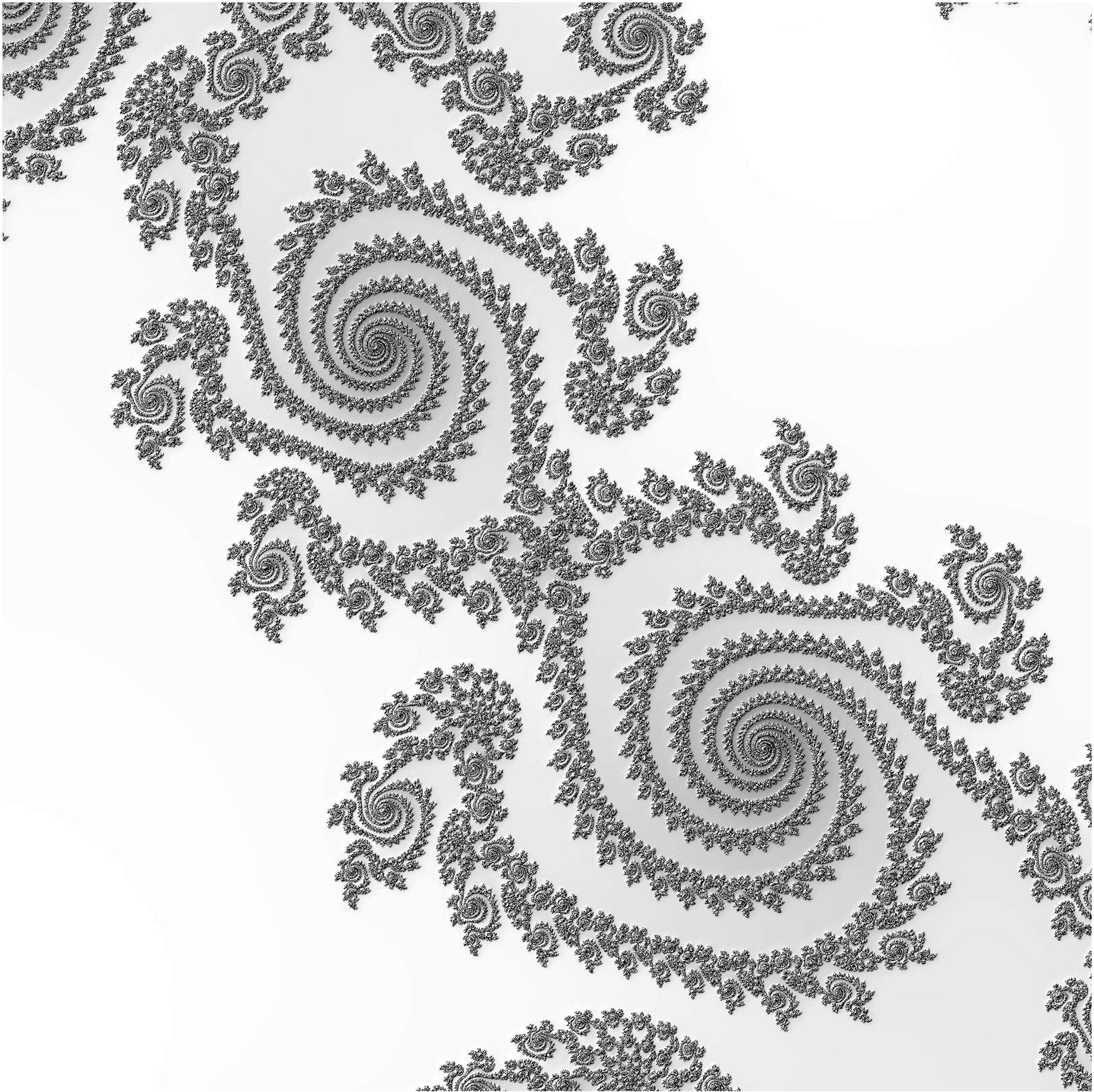


Tertiary lobes

A tertiary lobe is a lobe on a lobe on a lobe (on the main lobe). For example this is the neuron of lobe 4:5:6, that is to say, the 6th primary lobe on the 5th primary lobe of lobe number 4:



The principal synapse is the one closest to the lobe and you can just make out that it has 6 branches. Then there is a long serpent like section which eventually leads to a large synapse with 5 branches each of which end in a spiral synapse with 4 branches. Interestingly, between these primary synapses there are several prominent order 2 spiral synapses giving the neuron its serpent-like structure and which owe their existence to lobes 4:5:6:2 and 4:5:6:2:2 and 4:5:6:2:2:2 etc. which are always present.



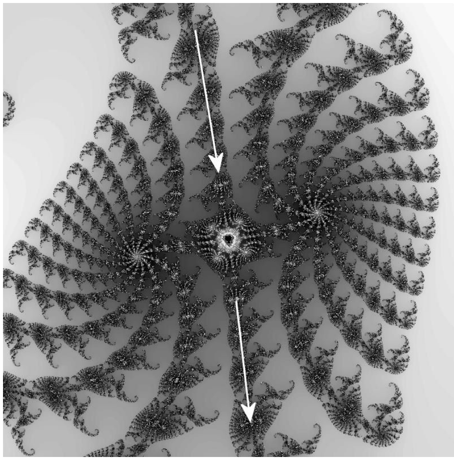
Axons and Dendrites

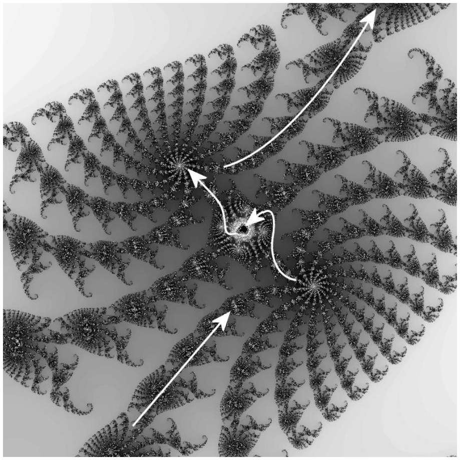
Every synapse has one axon and several dendrites. The axon is ultimately connected to the main lobe while the dendrites terminate in a synapse of order 1. You can tell the difference between an axon and a dendrite because the minibrots on the axon point *towards* the synapse while the minibrots on the dendrites point *away* from the synapse. You can see this clearly when the order is small but it is not always so easy to see the difference when the order of the lobe is large. The illustration on the opposite page shows the principal synapse of lobe 13. Can you see which of the 13 branches is the axon?

At first glance, all the branches look pretty much identical but a close up look at the two 'eyes' near the top of the image reveals subtle differences.

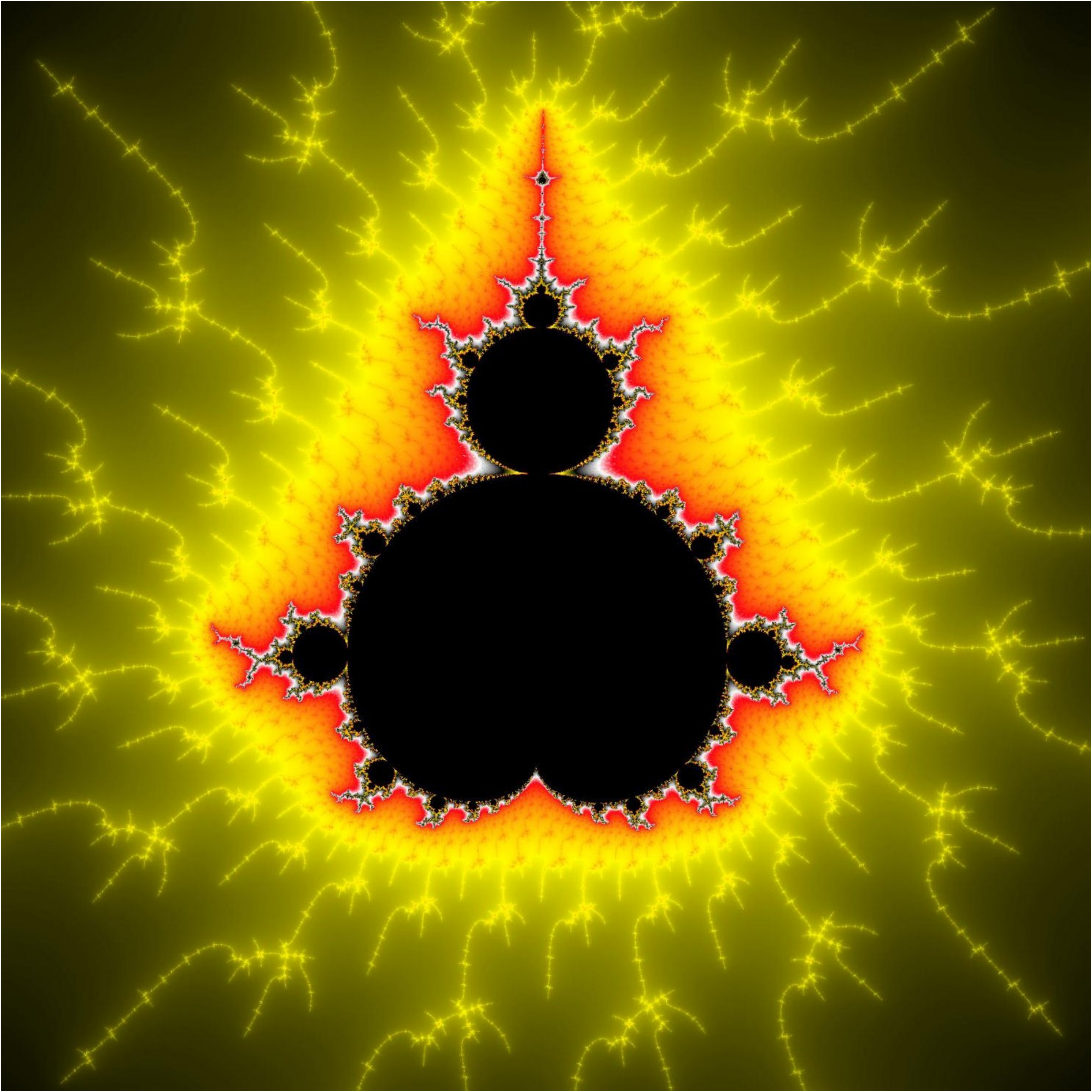
In the upper image, the axon flows straight down through the minibrot as indicated by the arrows but in the lower image, the 'axis' of the dendrite follows a zig-zag path up from the bottom and out at the top right hand corner.

Another way of spotting the axon is to look for the eye which has its spiral synapses perpendicular to the axis of the branch.



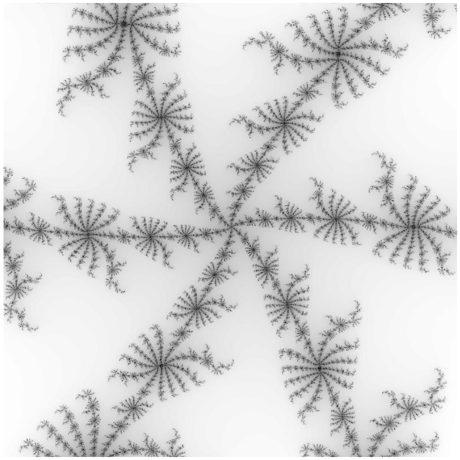


Shiva the Goddess
in the tendrils of lobe 4



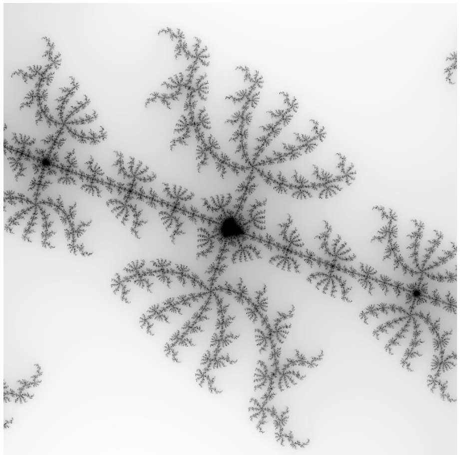
Principal synapses

Now we have oriented ourselves with regard to the main map, where shall we start looking for some interesting features? Let's start with the principal synapses. Here are close views of the principal synapse of lobe 6 which we might call the 'snowflake synapse'.



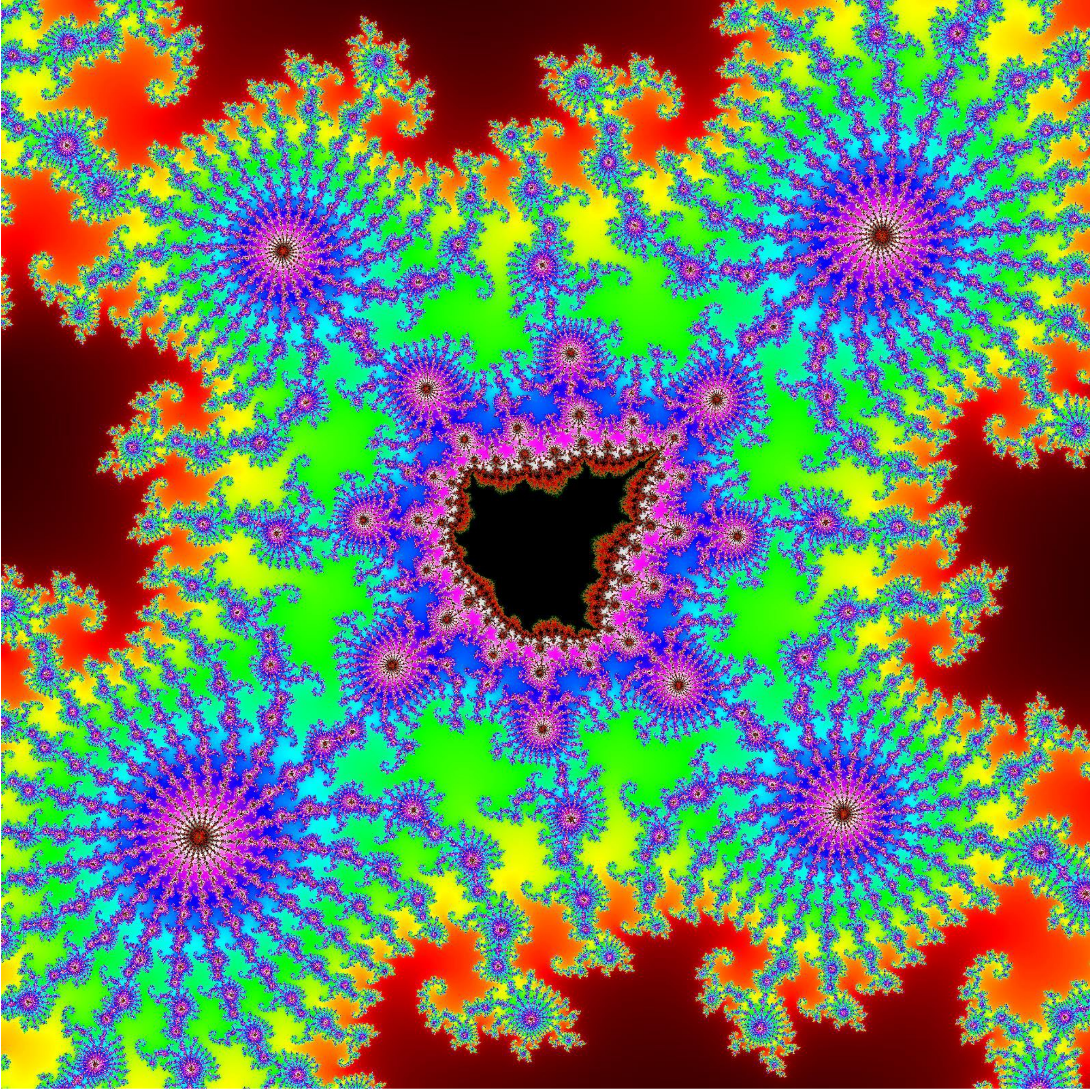
All synapses are self-similar – that is to say, they look exactly the same however much you magnify them. You will, however, notice that on the arms of the synapse, there appear to be synapses with twice as many arms. These are not real synapses though and if you magnify them sufficiently you will find a minibrot at the bottom with two genuine synapses, one on each side.

The main neuron always passes through the minibrot along its axis and there are 2 synapses on each side. If the order of the lobe is N, each of these synapses will contribute $N - 1$ dendrites which, together with the ingoing and outgoing axon make a total of $2(N - 1) + 2 = 2N$ arms. We can now see why, from a distance, this can look like a synapse with twice as many arms.



The details of these 'eyes' are infinitely varied and the source of many published images. The coloured image opposite shows a close up of one of the 'eyes' of synapse 4›3^6.

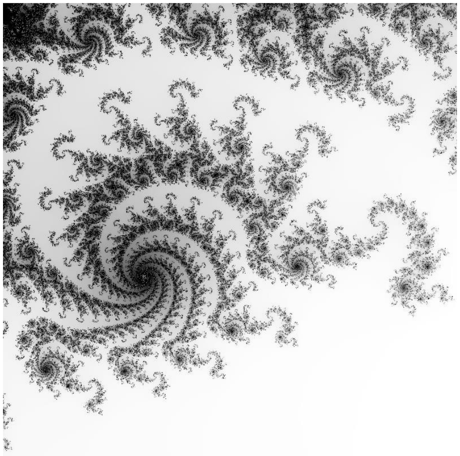
An 'eye' in the synapse of lobe 4›3^6

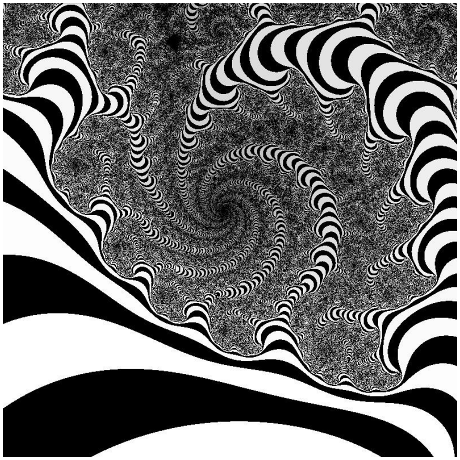


Spiral synapses

To find a spiral synapses with, say, 6 arms we need to look at the secondary lobes of lobe number 6. This one is on lobe 6/5. The first image is a smooth greyscale image while the second is exactly the same location but in binary black and white. The smooth areas between the spiral arms in the first image are revealed as huge roots which spiral down into infinite depths.

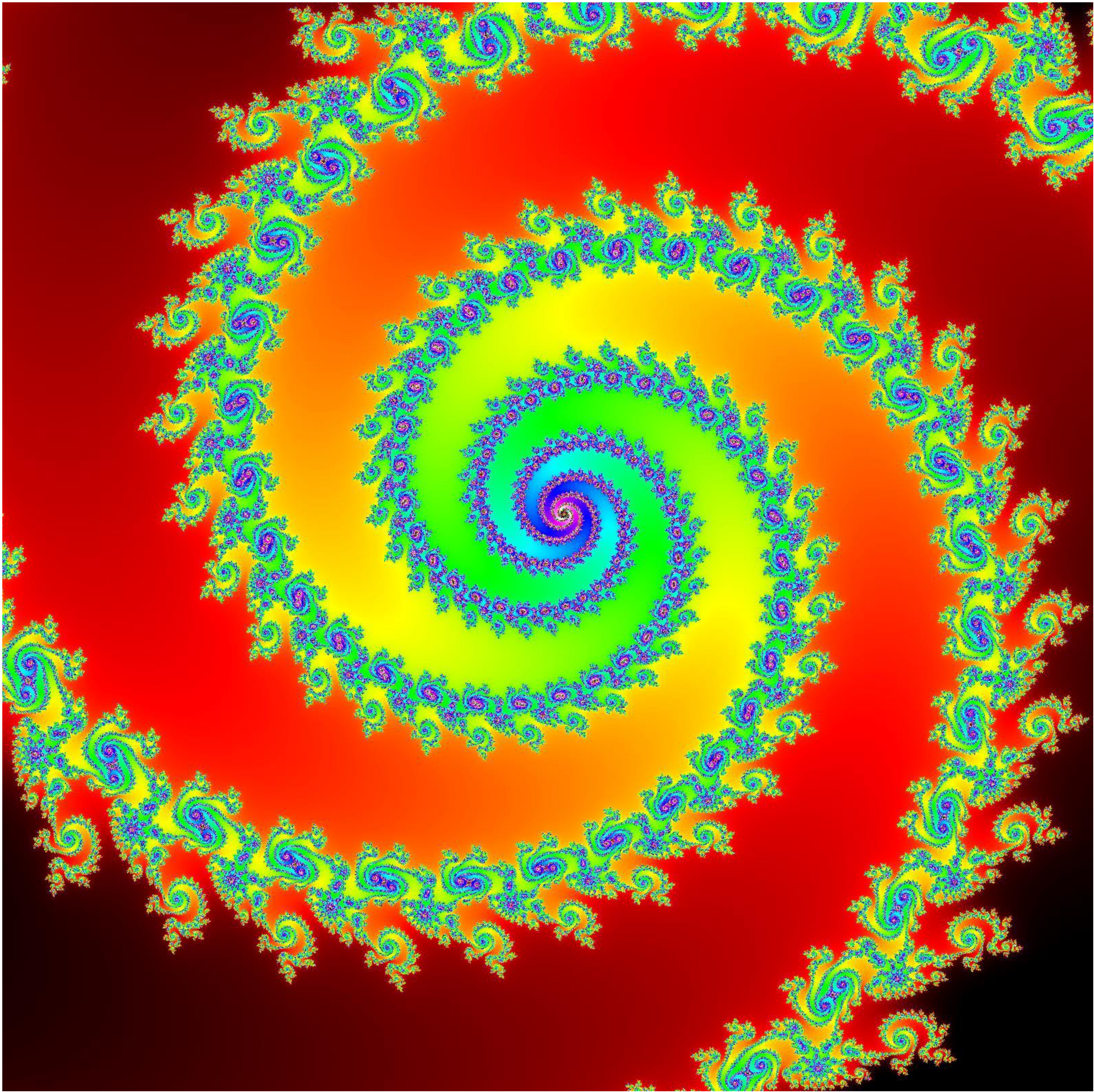
Compare that with the spiral synapse on lobe 2/6/5 (below) which has order 2 spiral synapses tacked onto every branch.



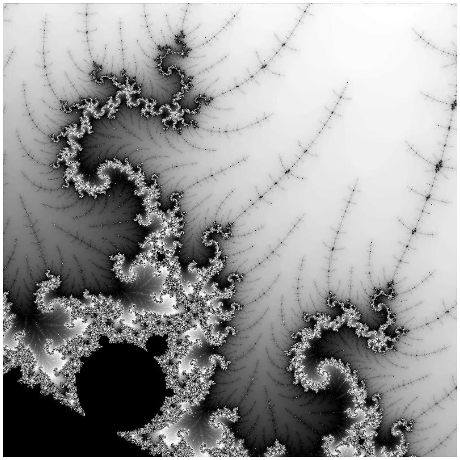


As with principal synapses, it is the 'eyes' of the synapse which hold the greatest interest. Here is the spiral synapse of lobe 3/5 and a close up of one of its eyes. Typically, each eye is bounded on each side by a spiral synapse of its own. The eyes increase in complexity as you go further down the main spiral.

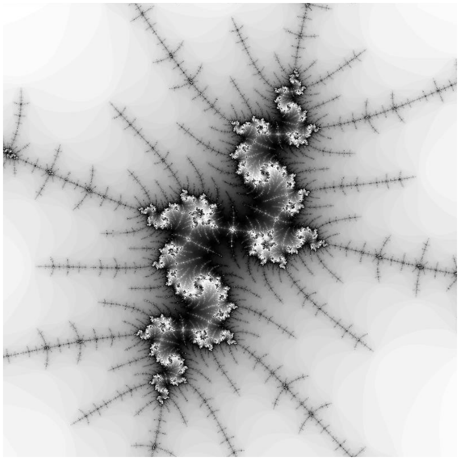
Spiral synapse of lobe 3:10



Dendritic minibrots



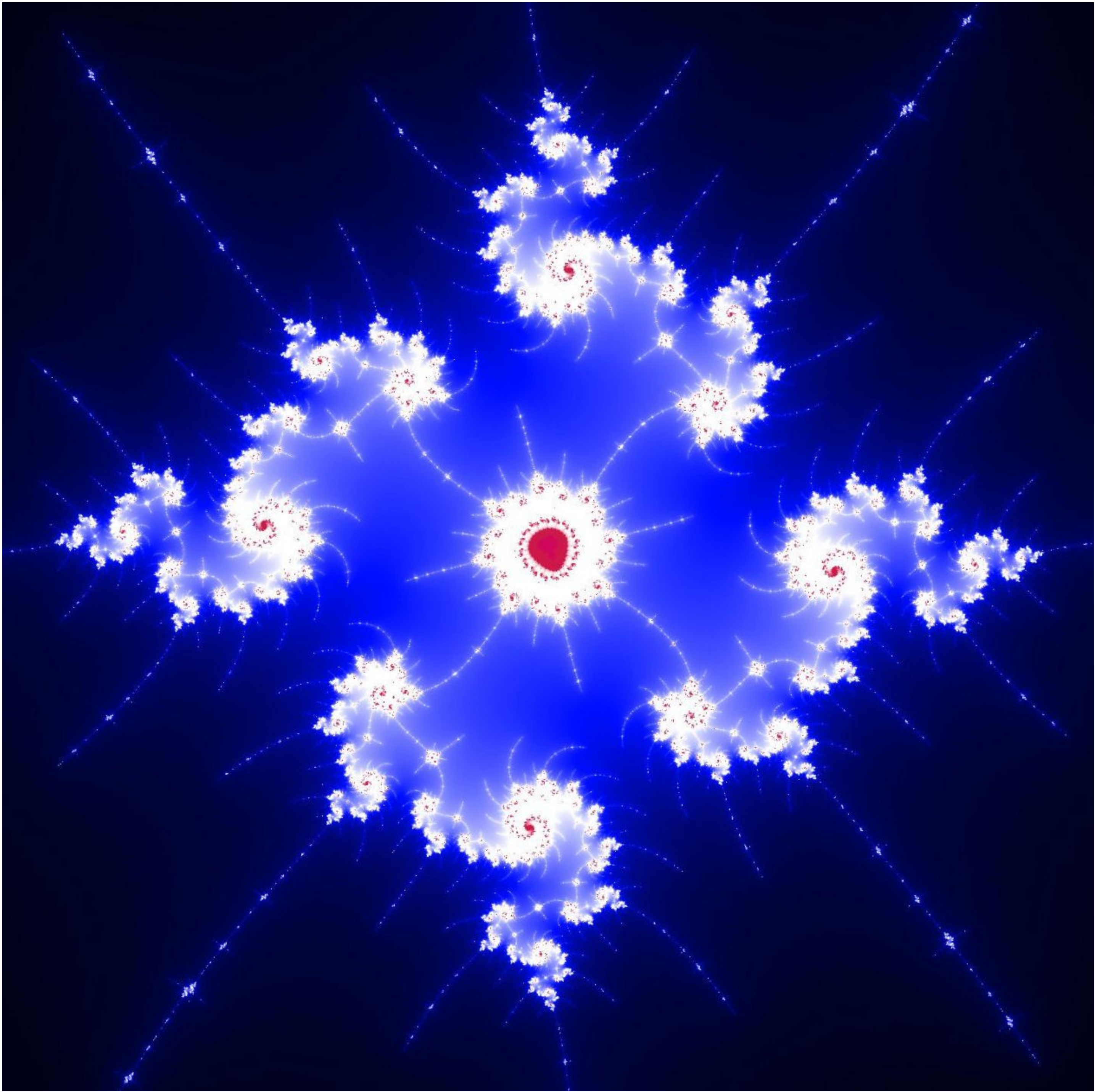
The image on the right is lobe 2/6 on the largest of the minibrots on the principal axis. You can tell that it is lobe 6 because of the principal synapse of order 6 and you can tell that it is on lobe 2/6 because of the order 2 spiral synapse. It is the long tendrils which tell you that this is not on the main brot but on one of the axonal minibrots. The image below is a close up of one of the blips on one of the tendrils.



If you have ever seen images of the closely related Julia sets you will probably recognise the general shape of this blip which does indeed look very much like the Julia set which corresponds to this point on the Mandelbrot map. The difference is this. If you blow up a Julia set, you always get more of the same. Every bit of it looks like the whole thing. This structure is different because it contains a minibrot at its centre which looks like a priceless jewel:

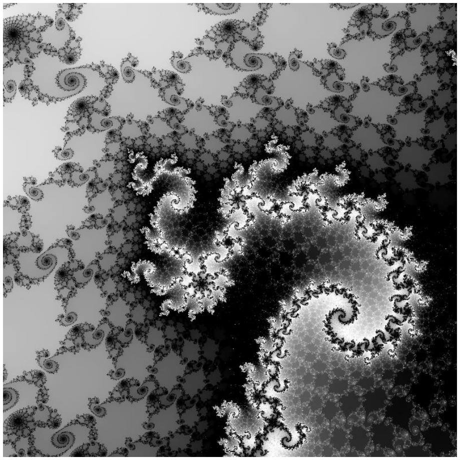
And what lady would not adore such a fabulous brooch!

Moreover, there are other gems waiting to be discovered elsewhere inside this structure. You could, in fact, write several pages on this structure alone but I will content myself with drawing your attention to the image on page 35 which must be a distant galaxy in another universe! (Incidentally, the magnification here is of the order of 2 billion so the odds of anyone else having seen this image before today are virtually zero.)



Asymmetrical features

While it is natural to want to zoom in on those features like minibrots and spirals which have a high degree of symmetry, it is worth seeking out places with other attractions. There is, for example, the procession of circus elephants illustrated on page 5 which can be found in 'elephant valley' – the cusp of the cardioid; the sea horses which are found in the crack between lobes 1 and 2 (see page 7), or the butterfly wings which can be found on the edge of lobe 4 (illustrated on page 13).

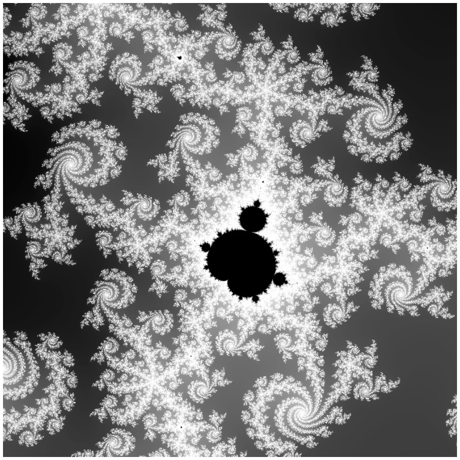


Double spirals can be particularly beautiful.

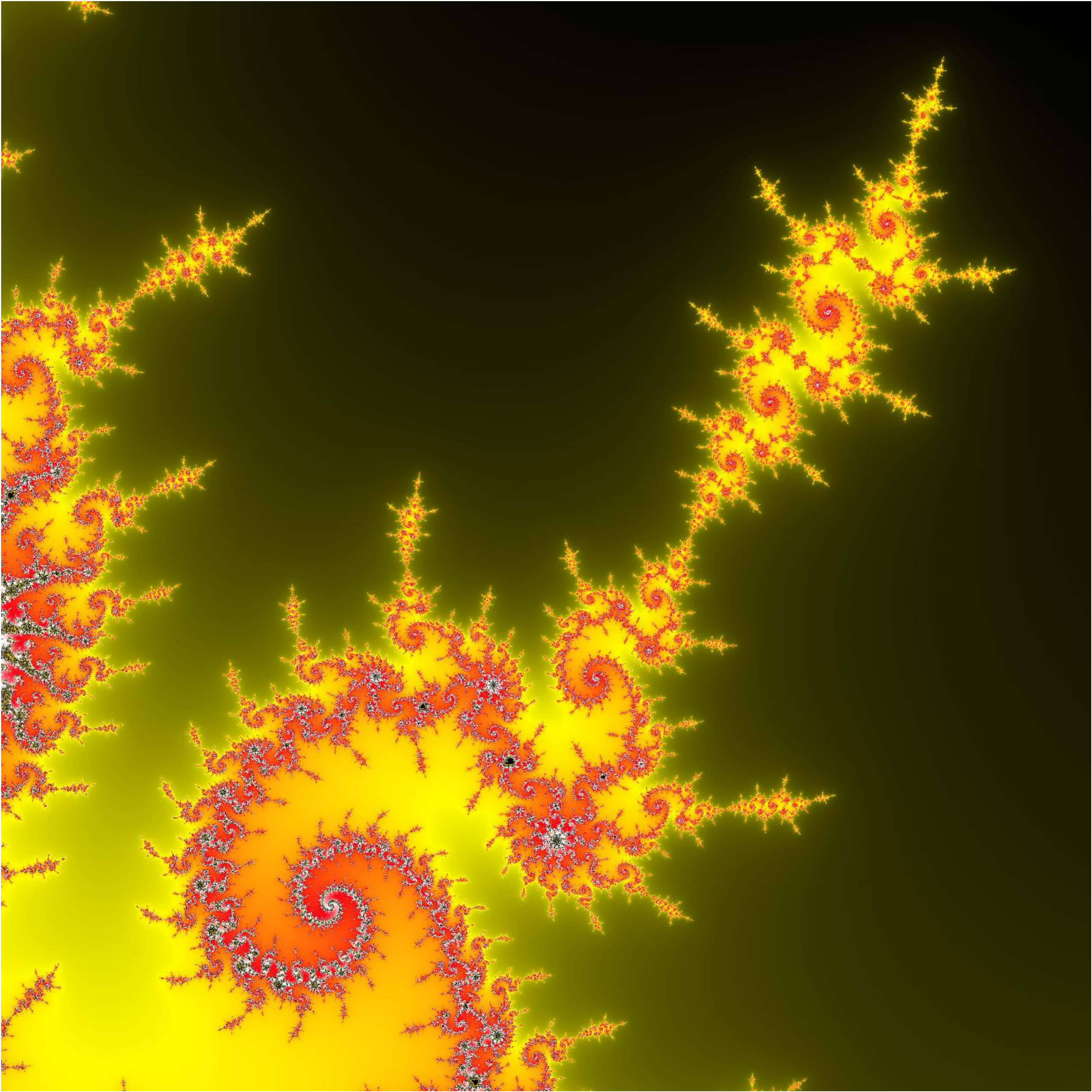
Another favourite image of mine is the candlestick bracket on lobe 2:9 illustrated on page 9.

All of these features can be found in subtly modified forms. For example the candlestick bracket on lobe 2:2:9 is transformed into a fire-breathing dragon!

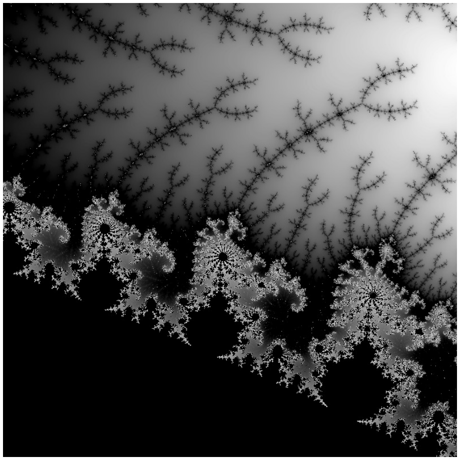
Most minibrots are perfectly formed but some are curiously misshapen Take this one, for example, which can be found in the principle dendrite of lobe 6:2›1[3]



Fire-breathing dragon on lobe 2:2:9

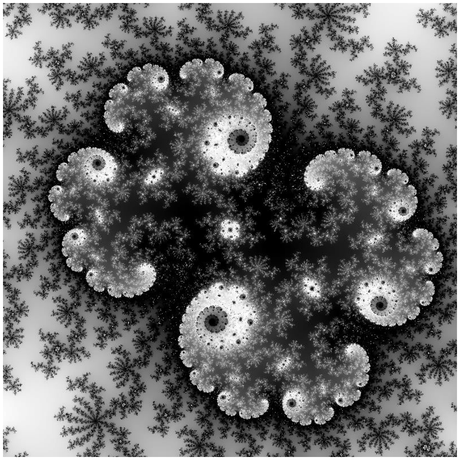


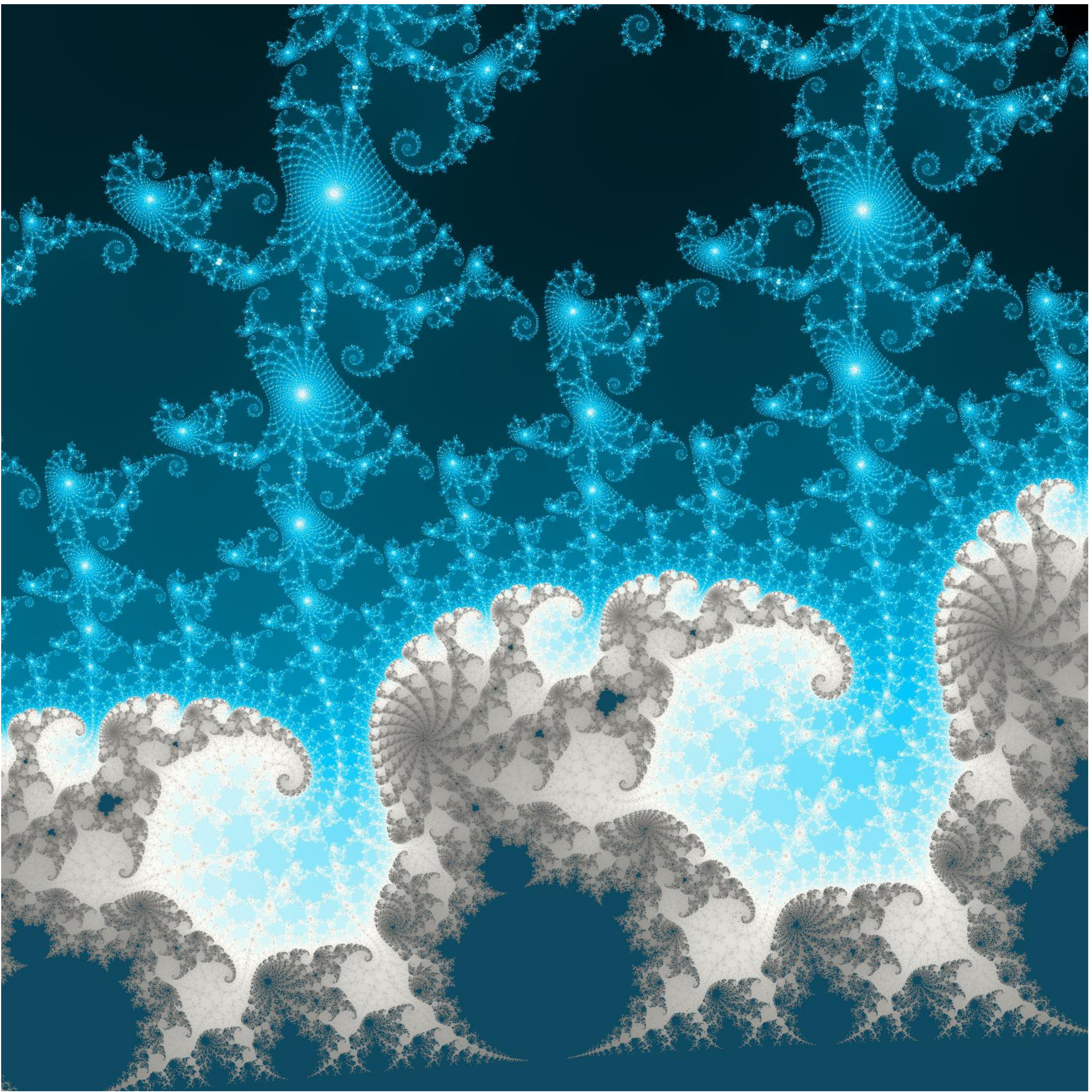
I have already noted that, unlike Julia sets, the Mandelbrot map is not self-similar. The deeper you go you will find new structures appearing. It is, however, not easy to know where to start delving in search of new structures. It is all too tempting to zoom into a spiral synapse, for example, hoping to find something new only to find that it goes on for ever and ever. Or you might zoom into a minibrot only to be disappointed when you only find elephants and sea horses there. Mind you, the elephants and seahorses are not quite the same. These elephants are to be found in the principal minibrot of lobe 3:



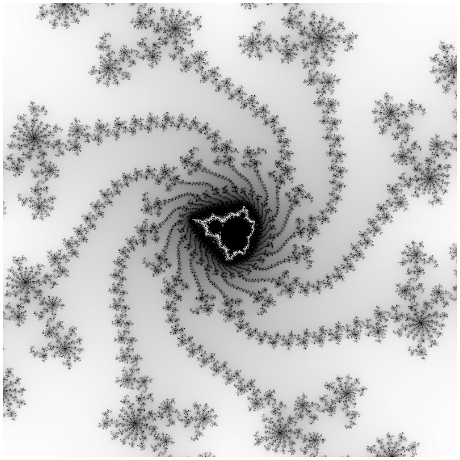
You will notice that they have both sprouted 'hair' with characteristic order 3 synapses. In addition, the 'hair' is decorated with beautiful brooches similar to but not the same as the ones that we found on the minibrots on the main axon.

If you choose any minibrot in one of the seahorses in seahorse valley (e.g. lobe 3.2^3) and then examine its elephants you will discover that the elephants are surrounded by a swarm of seahorses! Also you might care to examine the cauliflower heads in amongst the seahorses.





Delving deeper



It is often a good idea simply to dive in deep, deliberately avoiding the obvious minibrots and synapses. If you are lucky you can find structures which are not seen at higher levels.

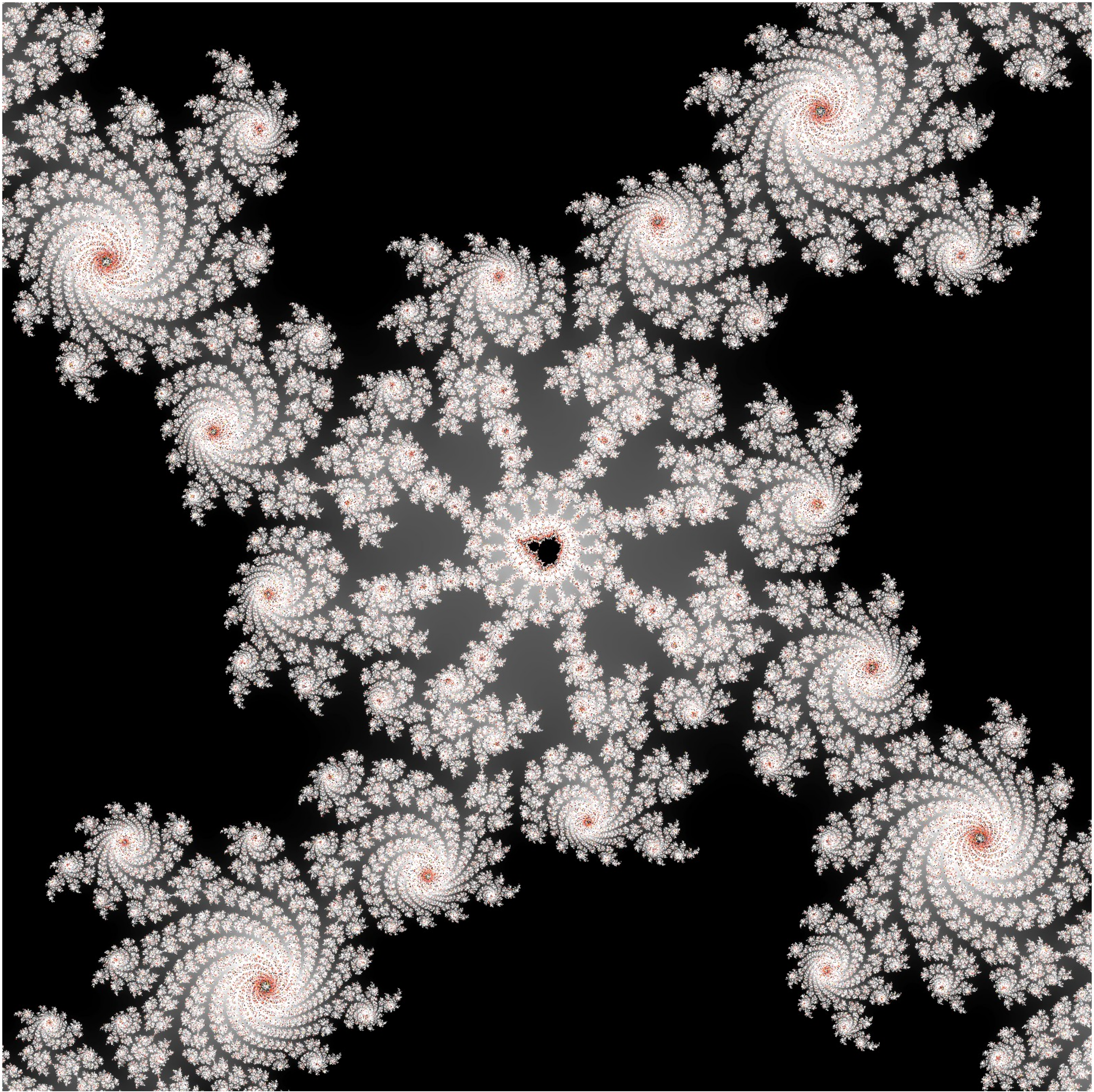
The minibrot in the upper illustration on this page is right at the end of the principle dendrite of lobe 3›2^4. Its coordinates are

(-0.722316754550125, 0.299224925775106)

and the magnification is about 400 million.

The image opposite is one of the linking minibrots in the neuron attached to lobe 6›5:2›1^3 at a magnification of about 4 billion. At this scale, the complete Mandelbrot set would be about a light year across.

Catherine wheel minibrot



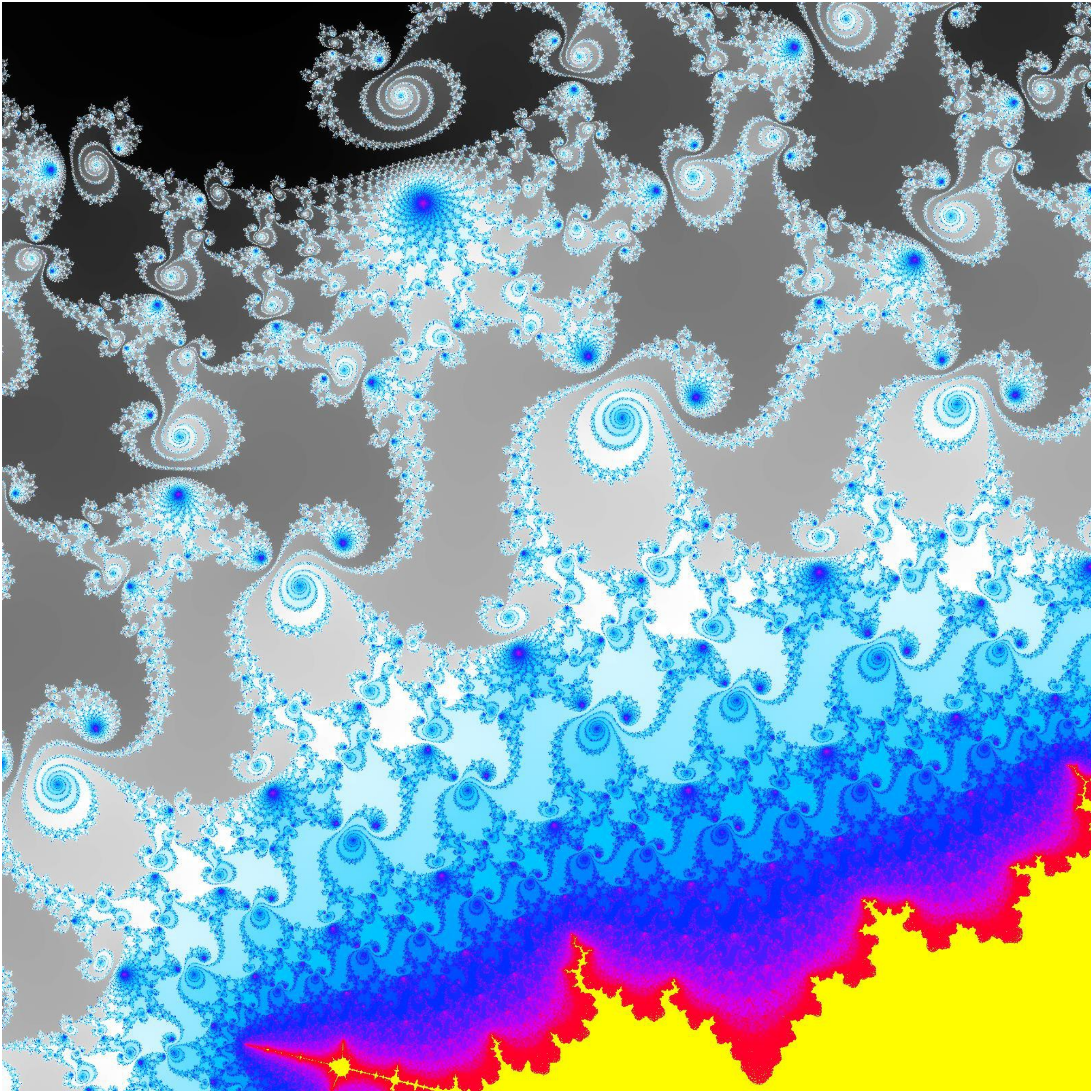
Coordinate Data

Page 3: Elephant Julia Set
C = (0.271787, -0.005557) × 1
Page 5: Elephant in Elephant Valley
(0.271787, -0.005557) × 1150
Page 7: Sea Horse in Sea Horse Valley
(-0.741444, 0.168525) × 150
Page 9: Candlestick bracket
(-0.781100, 0.141529) × 1840
Page 11: Banded minibrot
(-0.087465, 0.962527) × 393 thousand
Page 13: Butterfly wing
(0.356693, 0.352001) × 2000
Page 15: China plate
(-0.160713, 1.036997) × 17 million
Page 17: Nuclear explosion
(-1.749081, 0) × 197 thousand
Page 19: Forest fire
(-1.377177, 0.023304) × 160
Page 21: Spiral synapse in Sea Horse Valley
(-0.039879, 0.682584) × 768
Page 23: Lightning bolt
(-1.403593, 0.029453) × 768
Page 25: Deep minibrot in lobe 3
(-0.101776751, 0.950035003) × 174 million
Page 27: Peacock synapse
(-0.744009, 0.148175) × 4270
Page 29: Pretty wallpaper
(-1.112067,0.228032) × 393 thousand
Page 31: Axonal minibrot
(-1.543713212, 0) × 200 million
Page 33: Wedgwood plate
(-0.063756, 0.663692) × 34 thousand

Page 35: Spiral galaxy
(-1.769737367, 0.004811598)
× 201 million
Page 37: Embossed plasterwork
(0.274956, 0.484571) × 340 thousand
Page 39: Shiva the Goddess
(-1.744452, -0.022024) × 164 thousand
Page 41: Eye in lobe 4/3^6
(-0.043555, 0.653373) × 158 thousand
Page 43: Principal Spiral of Lobe 3.10
(-0.063854, 0.665319) × 6144
Page 45: Diamond and Ruby ring
(-1.769735, 0.004819) × 2180 thousand
Page 47:Fire dragon
(-1.262349, 0.045676) × 550
Page 49: Elephants in Sea Horse Valley
(-0.748960938, 0.0884845226506392)
× 25 million
{age 51: Catherine wheel minibrot
(0.374138249041388, 0.27261666562)
× 4 billion
Page 43: Minibrot in Sea Horse Valley
(-0.755408688, 0.059108588)
× 400 million

Back cover: Spiral of spirals
(-0.05712506, 0.66717175) × 100000

The edge of a minibrot somewhere in sea horse valley



All the images in this article were generated by a computer program called 'Mandelbrot Explorer' written by me and available from my website at: http://www.jolinton.co.uk/software.html