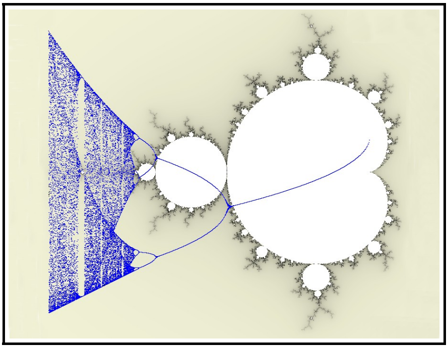


Chaos

and the Logistic Equation

Oliver Linton

CONTENTS

THE LOGISTIC EQUATION

In 1976 Robert M. May, a biologist from Princeton University published a paper in Nature titled "Simple mathematical models with very complicated dynamics" in which he analysed the following innocuous-looking equation:

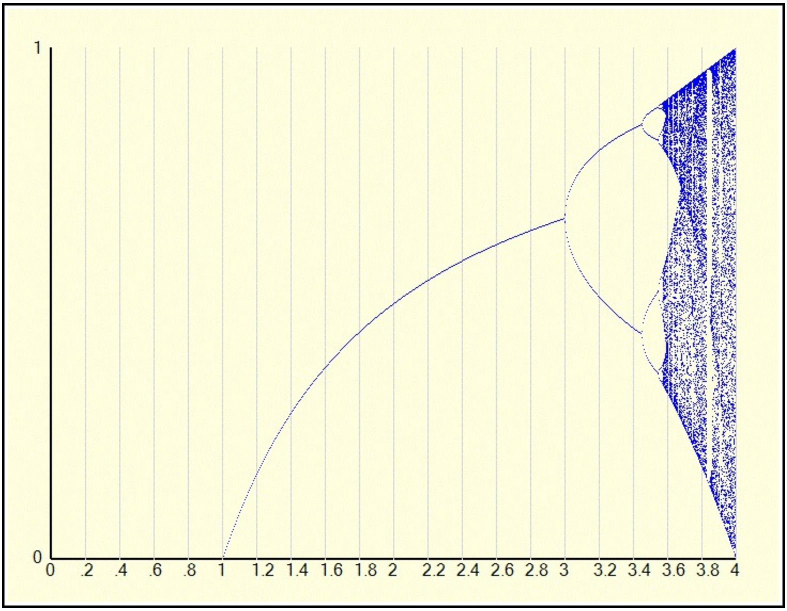
$$f^1(A, x) = Ax(1 - x) \tag{1}$$

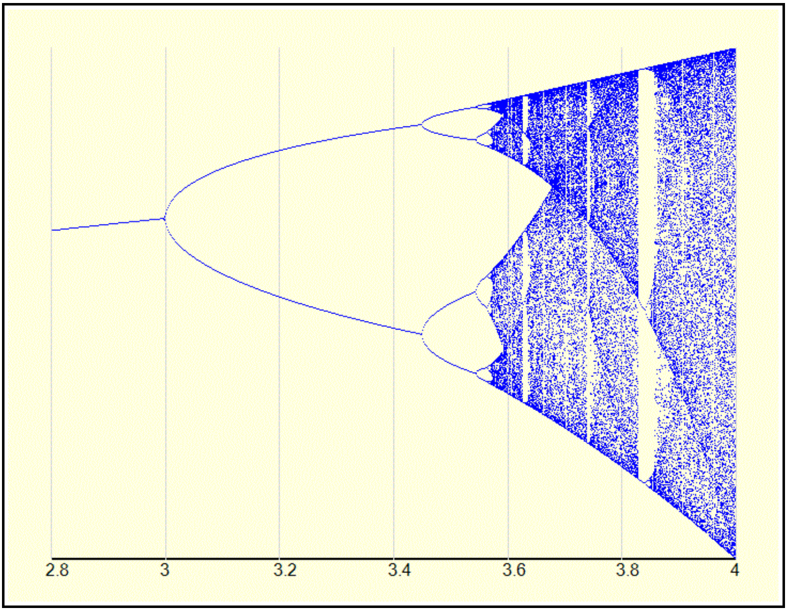
The idea is this. A is a constant in the range 0 – 4 while x is a variable. You start by putting x equal to some initial value x_0 and then calculate $x_1 = f^1(\mathrm{A}, x_0)$; then you do the same with x_1 calculating x_2, x_3 etc. etc. For some values of A, x homes in on a stable value more or less quickly but for other values of A x skips about apparently at random.

The behaviour can be illustrated by the famous bifurcation diagram shown opposite which plots many values of x (on the vertical axis) against A (on the horizontal axis.

Several questions immediately present themselves:

1. What is the shape of the stable region between 1 and 3?
2. Why does the graph split at $a = 3$?
3. Why does the graph split again at approximately $a = 3.45$?
4. Why do both branches split at exactly the same point?
5. What causes the prominent gaps in the chaotic region?
6. What are the equations of the boundary lines which border the chaotic region?
7. What causes the shadowy lines which cross the chaotic region?
8. What causes the chaos anyway and is it really random?
9. Do other functions produce other maps?


1
0
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4


2.8
3
3.2
3.4
3.6
3.8
4

Other Functions

Lets deal with the last question straight away.

Opposite are the graphs several functions together with the chaos maps which they produce when iterated.

Graph A is the logistic equation (1) but with A values from -2 to 4. The region with negative values of A is a distorted version of the positive map and is usually ignored.

Graph B is a cubic equation constrained within the same x limits of 0 and 1. Unlike the logistic equation, it is symmetrical about the point (0.5, 0.5) and its chaos map reflects this symmetry. It shows all the main features of the Logistic Map but, surprisingly, it appears to be missing one half of the second bifurcation (at $A = 3$). (This is due to the fact that it has two stationary values rather than just one.)

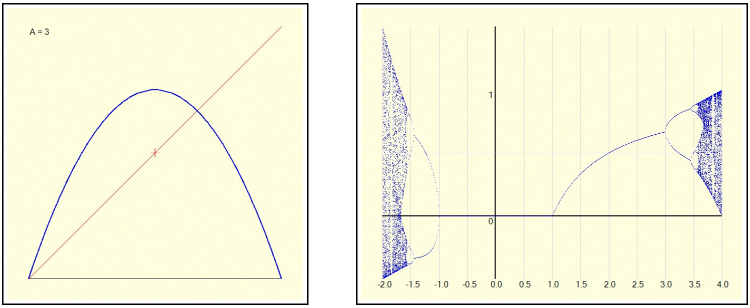
Graph C is the function

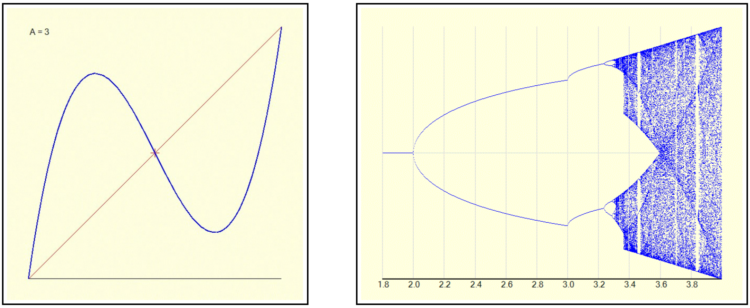
$$f^1(A, x) = x^2 + A$$

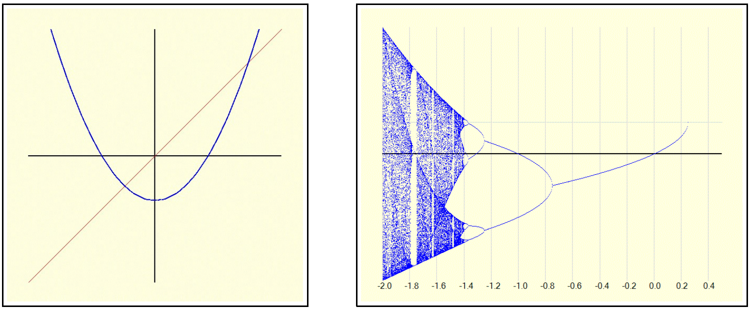
and is the same one as the one used to generate the Mandelbrot map. It produces a map which is essentially the same as the Logistic Equation except that it is upside down.

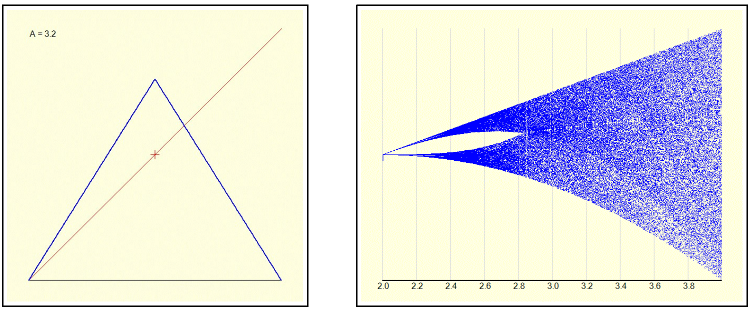
Graph D is called the tent function and consists of two straight lines with gradient $A/2$ and -A/2. Owing to the fact that it is not differentiable at the apex, it does not show the classic bifurcation – basically all the period doublings happen simultaneously. In addition, there are no bands of stability. Even so, it shows several features in common with the Logistic Equation.

Essentially, whatever function you choose will do – provided it has at least one maximum or minimum. Traditionally the logistic equation has been the one subject to most study and we shall go along with that.


A = 3
1
0
-2.0 -1.5 -1.0 -0.5 0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0
A


A = 3
1.8 2.0 2.2 2.4 2.6 2.8 3.0 3.2 3.4 3.6 3.8
B


-2.0 -1.8 -1.6 -1.4 -1.2 -1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4
C


A = 3.2
2.0 2.2 2.4 2.6 2.8 3.0 3.2 3.4 3.6 3.8
D

ATTRACTORS AND REPELLORS

In order to explain how what is going on, lets start with a slightly simpler system: *Think of a number x. Halve it and add 1. Repeat over and over again*. What happens? You will find that whatever number you started with, you will end up getting closer and closer to x = 2. This is an example of what I call a *progressive attractor.*

Now try the function *double x and subtract 2*. What happens now? Quite the opposite! Whatever number you start with (*except* x = 2) you end up at plus or minus infinity! Here x = 2 is a *progressive repellor.*

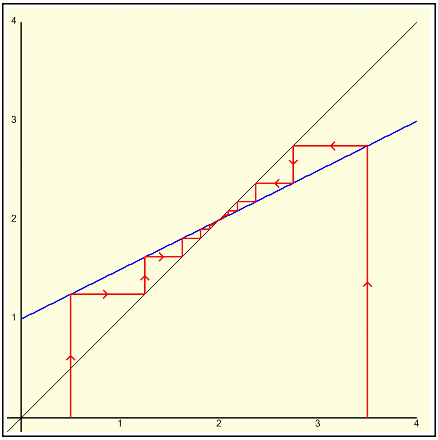
Why is there such a big difference between the two functions?

The crucial difference is that in the first case, the slope of the line ($y = x/2 + 1$) is less than 45º while in the second case ($y = 2x - 2$) it is greater than 45º . This is shown in the first two graphs on the opposite page. (The 45º line through the origin serves as a means of transferring the output of one calculation into the input of the next.)

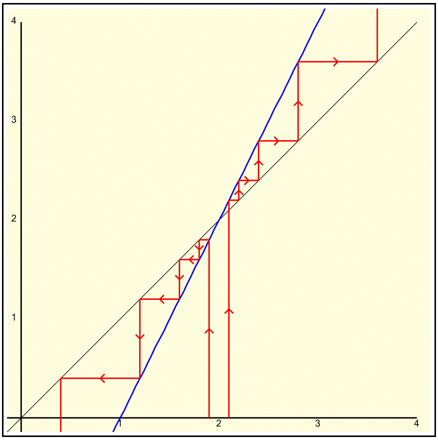
Lets try some *negative* gradients – i.e. lines which slope down.

The pattern is clear. If the gradient is shallower than 45º, the point is still an *attractor* but instead of *progressively* homing in on the stable point, it *cycles* towards it. If the gradient is steeper than 45º the point is a *cyclic repellor.* This is illustrated in the second two graphs.

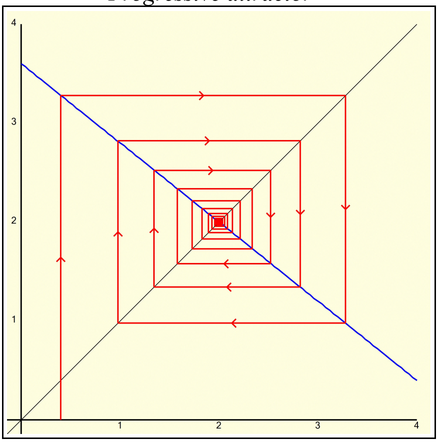
Something special happens when the gradient of the graph is zero; every initial starting point homes in on the stable value instantly.



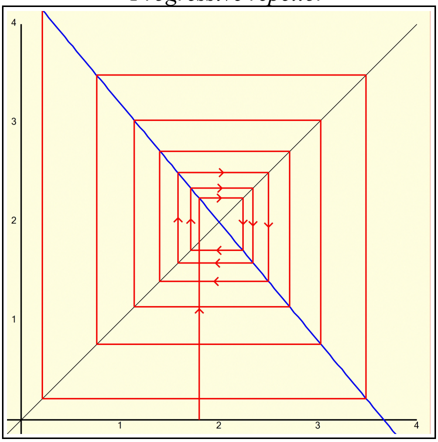
Progressive attractor



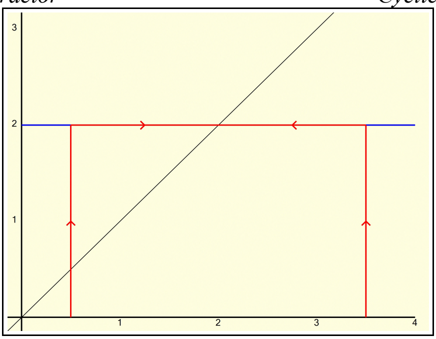
Progressive repellor



Cyclic attractor



Cyclic repellor



Critical attractor

THE STABLE REGION

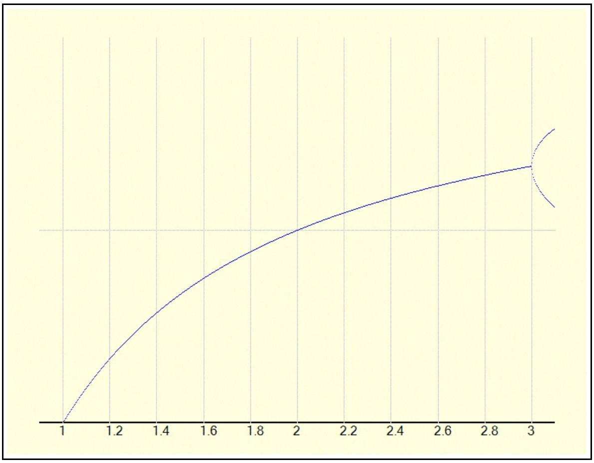
1 < A < 3

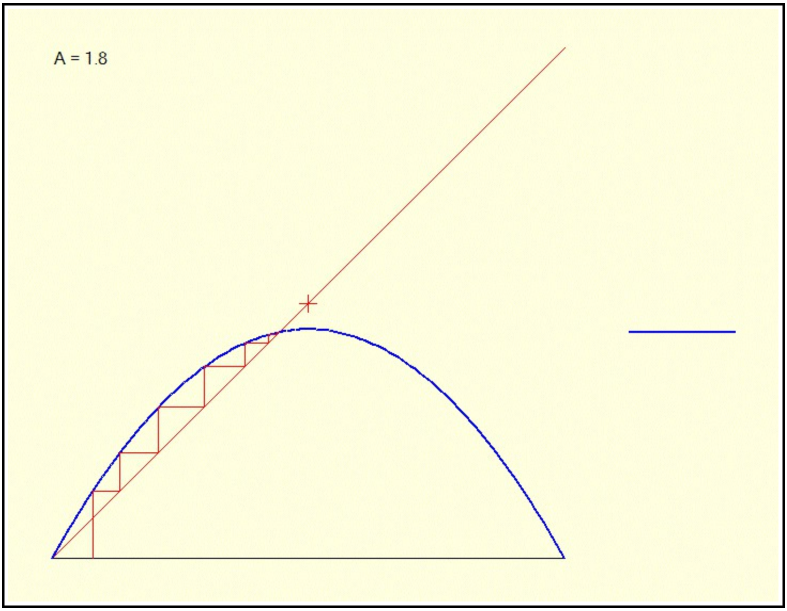
Now lets return to the logistic equation. The upper diagram shows what happens when $A = 1.8$. A small starting point zigzags its way up to the point where the function line crosses the 45° line. This is the place where $f^1(x) = x$ and this is the stable point for this value of A. We can easily calculate its value as follows:

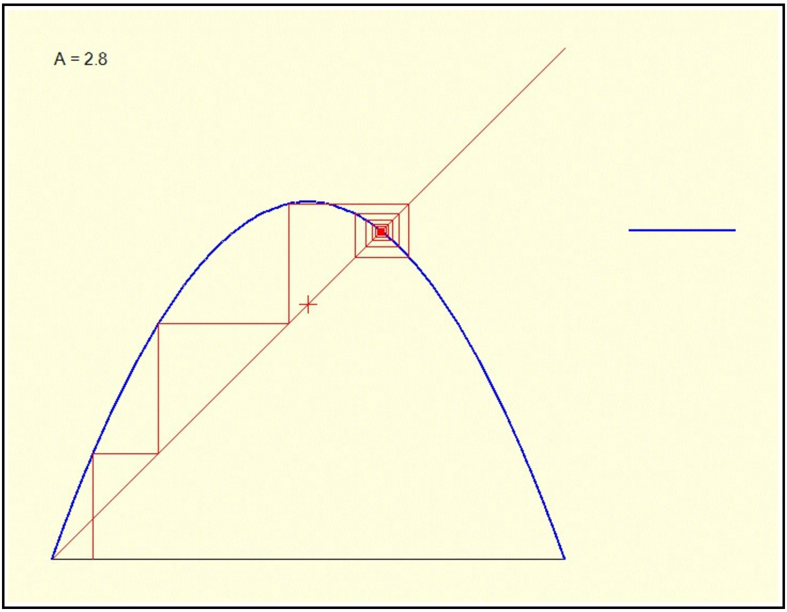
$$\begin{aligned} Ax(1-x) &= x \\ Ax - Ax^2 &= x \\ x = 0 \quad \text{or} \quad x &= 1 - 1/A \end{aligned} \tag{2}$$

There are two answers for x because the function crosses the 45° line in two places. At $x = 0$, the gradient of the function is greater than 1 and this point is a *progressive repellor*; at the second point the gradient is less than 1. This point is a *progressive attractor* and all starting points in the range 0 – 1 will end up there.

The lower diagram opposite shows what happens when $A = 2.8$. The gradient at the place where the curve crosses the 45° line is now negative but its magnitude is still less than 1 so it is still an attractor of the cyclic sort.




A = 1.8


A = 2.8

THE FIRST BIFURCATION

3 <= A < 3.45

At the point where $A > 3$, something very strange happens; instead of settling down to a single stable value, x oscillates between two values. You can see why. x tries to home in on the point where the function crosses the 45° line but when it gets there it finds that the (magnitude of the) gradient of the function is now greater than 1 and the point turns out to be a *cyclic repellor*. Fortunately, though, because of the curve on the function, x is able to find a pair of points where *average gradient*[1] is less than 1 and it is stabilises on these instead.

The point where the bifurcation starts can be calculated by finding the equation of the gradient of the function (obtained by differentiation) , putting $x = 1 - 1/A$ and equating this to –1.

$$\begin{aligned} grad &= A(1-2x) = A(1-2(1-1/A)) = -1 \\ A &= 3 \end{aligned} \tag{3}$$

If x undergoes a cycle of period 2 it is because two iterations of x get us back to where we started. i.e.

$$f^2(A,x) = f^1(A, f^1(A,x)) = x \tag{4}$$

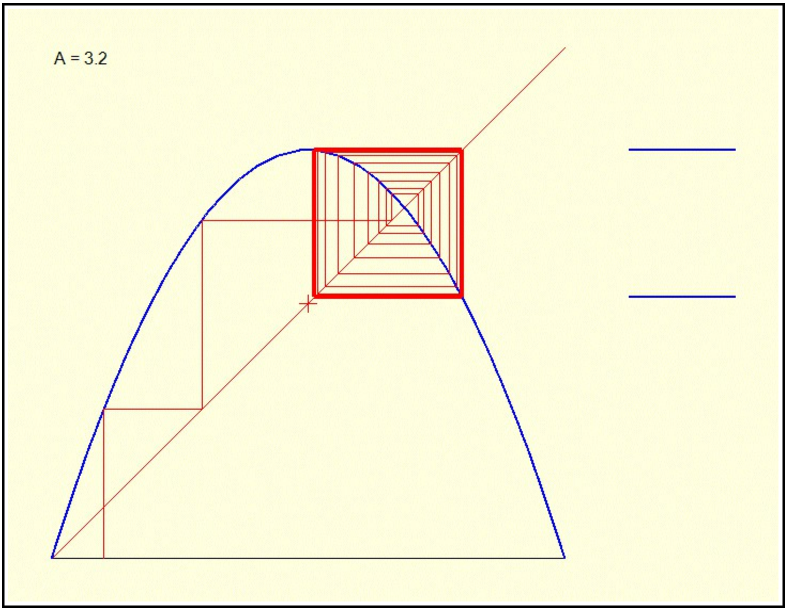
Now what does $f^2(A, x)$ look like? The equation looks pretty horrid:

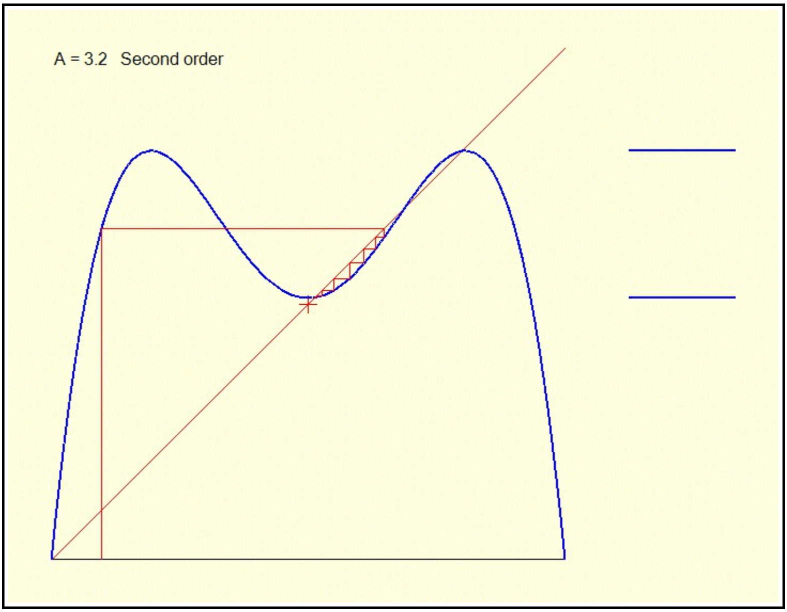
$$\begin{aligned} f^2(A,x) &= A(Ax(1-x))(1-Ax(1-x)) \\ &= -A^3x^4 + 2A^3x^3 - A^2(A+1)x^2 + A^2x \end{aligned} \tag{5}$$

On the cobweb diagram it looks like a twin humped camel which crosses the 45° line in three places (not counting the origin). Two of these points are attractors while the central one is a repellor.

Higher order f lines have more and more humps.

1 The average appropriate here is the geometric mean $\sqrt{G_1 G_2}$ (see page 38)


A = 3.2


A = 3.2   Second order

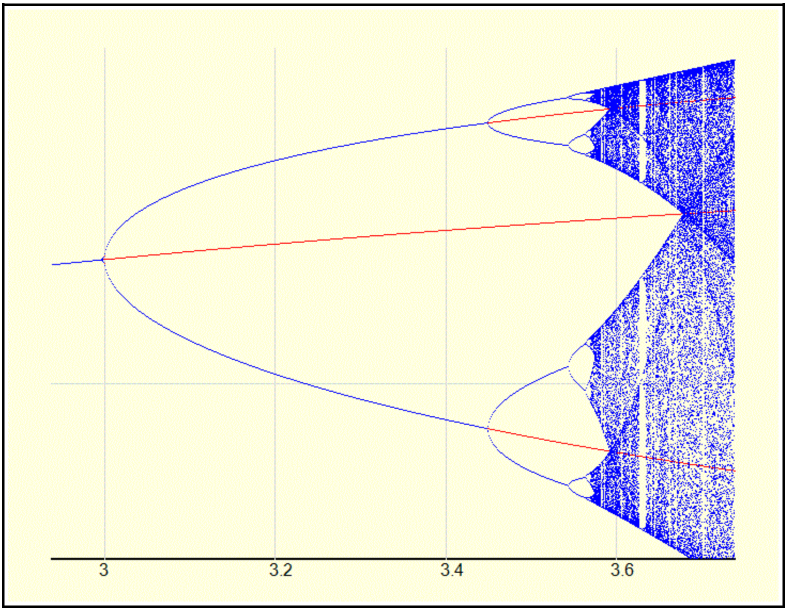
Lines Of Instability

Within the region $1 < A < 3$, there is always a starting value of x which is immediately stable, this value being $x = 1 - 1/A$ (equation (2)). Other starting values of x home in on this value more or less quickly.

When we move into the region where the map has split into two, between $A = 3$ and $A = 3.45$, this starting value becomes unstable because the magnitude of the gradient of f^1(A, x) (equation (1)) becomes greater then 1. Like a playing card balanced on its edge, the slightest deviation from the exact value will cause it to cycle further and further away from the line of instability.

Likewise, in the region $3.45 < A < 3.5$, there are two values inside the forks which are also unstable. These lines of instability are shown in the illustration opposite in red.

One interesting thing about these lines is that they pass right through another point of interest – the point where the regions of chaos overlap. We shall return to this point later.


3
3.2
3.4
3.6

The Second Bifurcation

$A = 1 + \sqrt{6} = 3.449489743...$

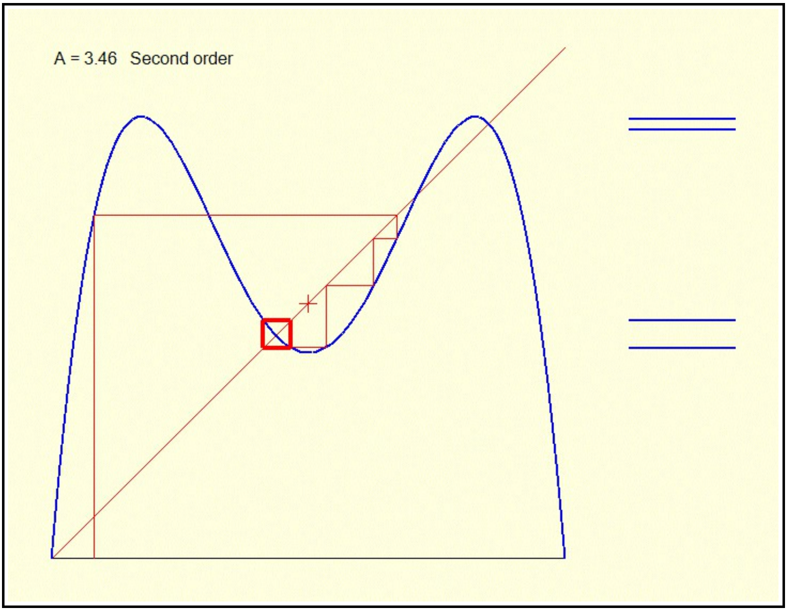
The actual values of the two points through which x oscillates during the first bifurcation period can be calculated by putting $f^2(\mathrm{A}, x)$ – equation (5) – equal to x and solving the resulting quartic equation for x in terms of A. This is not quite so difficult as it appears because we already know two of the solutions (0 and $1 - 1/A$)[2]. The other two solutions are:

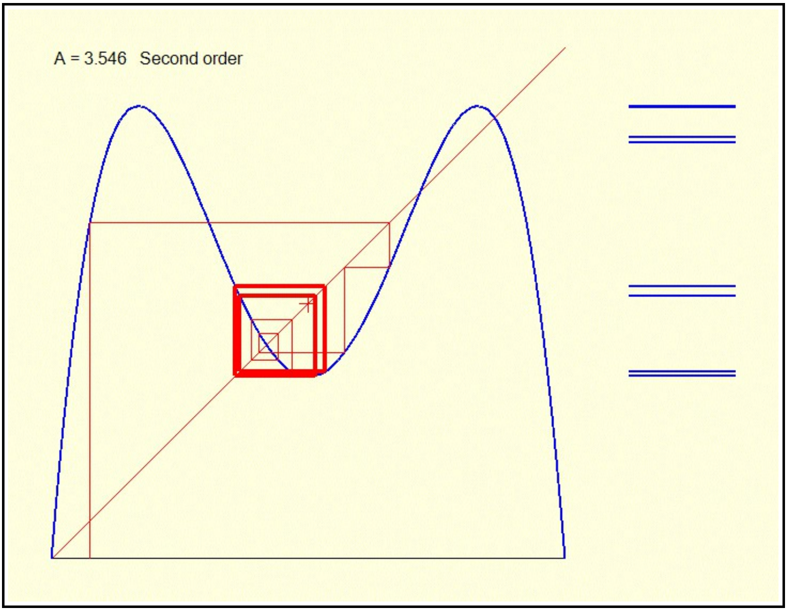
$$x = \frac{A + 1 \pm \sqrt{(A + 1)(A - 3)}}{2\,A} \tag{6}$$

The second bifurcation occurs when the gradient of the second order function reaches unity at the two attractors. Calculating exactly where this point is is not easy because the second order function (5) is already pretty complicated and the value of x which must be substituted into it is the one given in equation (6). Wikipedia informs me that the solution is $1 + \sqrt{6} =$ 3.449489743...

The lower graph shows what happens just beyond the third bifurcation point (A =3.527...) where the stable values split into 8. Increasing A further causes more and more rapid bifurcations until when A = 3.56995... (the Feigenbaum point, also known as the accumulation point of the first period doubling sequence) the chaotic region begins.

2 Equation 5 becomes $A^3 x^4 - 2A^3 x^3 + A^2(A + 1)x^2 - (A + 1)(A - 1)x = 0$
which factorizes into $x(Ax - (A - 1))(A^2 x^2 - A(A + 1)x + (A + 1)) = 0$


A = 3.46   Second order


A = 3.546   Second order

The Gradient Theorem

Now why do both branches bifurcate simultaneously? And why do they go on bifurcating simultaneously throughout the period doubling episode? The answer must be that the gradients of the higher order curves at the several attractors must be identical. There is a deep theorem here because *regardless of the function used* bifurcation *always* occurs at the same point however many branches there are.

We see that when $A = 3.43$, the map has split into two branches which oscillate between two values x_0 and x_1. Also shown on the graph is the function $f^2(x)$ at this value of A. We are interested in proving that the gradient of this curve is the same at these two values.

Using the chain rule for differentiating a function of a function we obtain:

$$\frac{df^2(x)}{dx} = \frac{df^2(x)}{df^1(x)} \times \frac{df^1(x)}{dx}$$

Now $\frac{df^1(x)}{dx}$ is simply the gradient of f^1 at the initial point $x = x_0$. Let us call this $G^1(x_0)$.

Also, since $f^2(x)$ is the iterate of $f^1(x)$, $\frac{df^2(x)}{df^1(x)}$ is the gradient of the f^1 curve at the first iterate of x_0: $G^1(x_1)$. i.e.

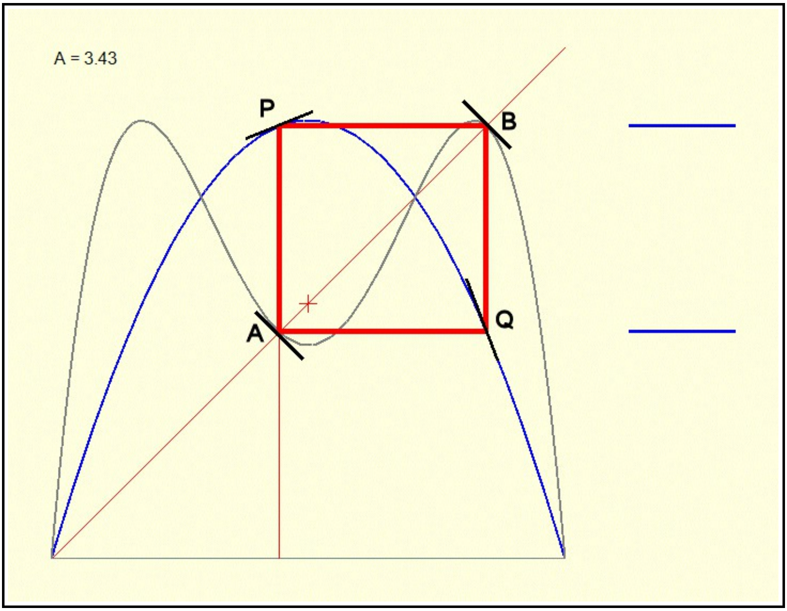
$$G^2(x_0) = G^1(x_1) \times G^1(x_0)$$

What this is saying is that on the diagram opposite, the gradient of the second order curve at A is equal to the product of the gradients of the first order curve at P and Q.

What about the gradient at B? This will be equal to.

$$G^2(x_1) = G^1(x_2) \times G^1(x_1)$$

But of course, for the second order curve $x_2 = x_0$ so the gradient at B is equal to the gradient at A.


A = 3.43
P
B
A
Q

The Gradient Theorem (2)

We can take this idea further.

$$\frac{df^3(x)}{dx} = \frac{df^3(x)}{df^2(x)} \times \frac{df^2(x)}{dx} = \frac{df^3(x)}{df^2(x)} \times \frac{df^2(x)}{df^1(x)} \times \frac{df^1(x)}{dx}$$

or

$$G^3(x_0) = G^1(x_2) \times G^1(x_1) \times G^1(x_0)$$

$$G^4(x_0) = G^1(x_3) \times G^1(x_2) \times G^1(x_1) \times G^1(x_0)$$

and so on.

When we apply this result to the fourth order *f* curve, we can see that the gradients at all the four points A, B, C and D are equal because they are all equal to the product of the gradients at P, Q, R and S.

The illustrations opposite below show what the first four *f* functions look like for a fixed value of A.

In each case we see that the each *f* curve has zero gradient (i.e. a maximum or a minimum) where either the previous curve has zero gradient or where the previous curve is equal to 0.5. But what is so special about $f^n(x) = 0.5$?

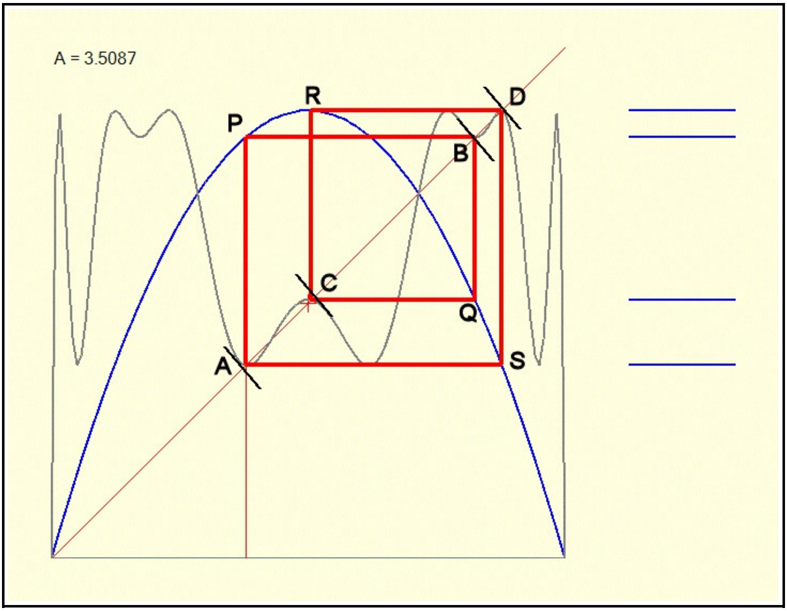
The answer is, of course, that this is precisely the place where the gradient of the f^1 curve is zero.

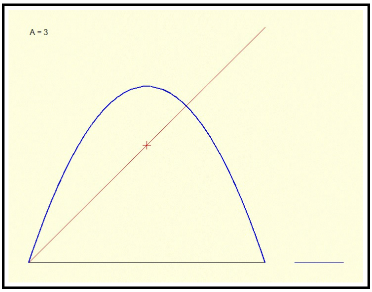
We can summarise all this as follows:

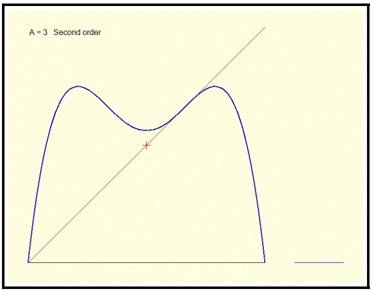
1. If the gradient of the f^n curve is zero at x, then the gradient of the f^{n+1} curve will also be zero at x.

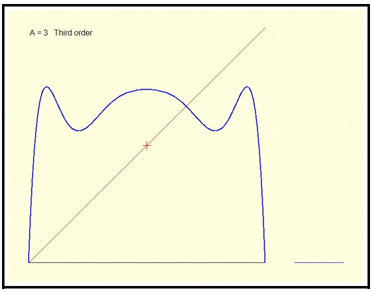
2. The gradient of the f^{n+1} curve will also be zero at the points where $f^n(x) = 0.5$ because the f^1 curve has zero gradient at $x = 0.5$.

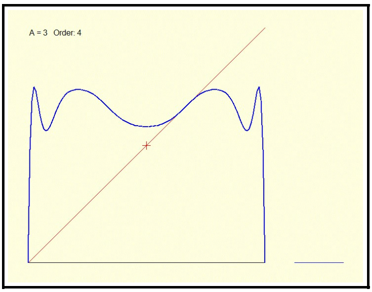
3. All the *f* curves have zero gradient at $x = 0.5$.


A = 3.5087
R
D
P
B
C
Q
A
S


A = 3


A = 3 Second order


A = 3 Third order


A = 3 Order: 4

THE CRITICAL VALUE

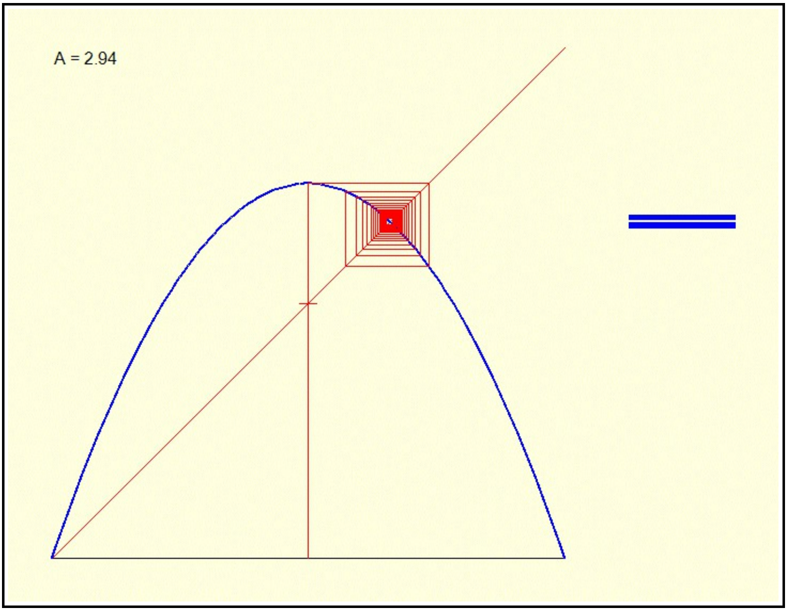
$x_0 = 0.5$

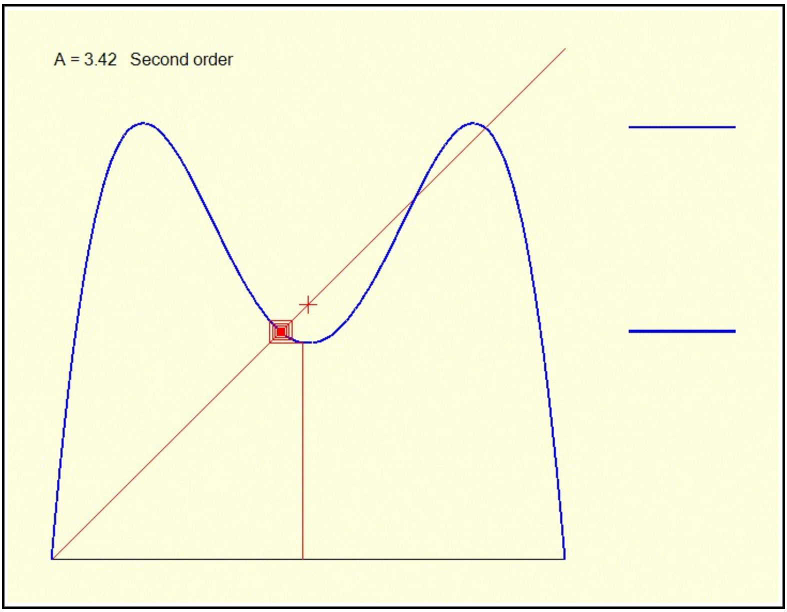
The third consequence is particularly important – *All the f curves have zero gradient at* $x = 0.5$. This point is therefore special and is called the *critical point* x_c. It is the point where the f^1 curve has zero gradient and we can calculate x_c by differentiating the logistic equation and equating to zero:

$$f = Ax(1-x) = Ax - Ax^2$$
$$\frac{df}{dx} = A - 2Ax = 0$$
$$x_c = 0.5$$

If you start with $x_0 = x_c$ then x will follow a certain set of specific values called the *critical orbit* of the function. From now on, we shall always use $x_0 = x_c = 0.5$.

Another reason for doing this is that if there is a stable cycle nearby, starting at the critical point will always find it. The upper illustration shows the critical value homing in slowly on a single stable point with A slightly less than 3. The lower illustration shows it finding a second order cycle.


A = 2.94


A = 3.42 Second order

SUPERSTABLE POINTS

β_0, β_1, β_2, etc.

According to a theorem by Pierre Fatou, within every stable region (e.g.1 $A < 3$) there will always be a point where the critical value of x is a member of the cycle – i.e. a point where x_c does not need to 'home in' on the stable cycle because it is already there.

In the case of the stable region between 1 and 3 we can calculate this value as follows:

$$f^1(A, 0.5) = A \times 0.5(1 - 0.5) = A/4 = 0.5$$
$$A = \beta_0 = 2$$

This value of A is called a superstable point and I shall call it β_0.

Between $a = 3$ and $a = 3.45$ (i.e. the region of the first bifurcation) we have:

$$f^2(A, 0.5) = A \times A/4(1 - A/4) = 0.5$$
$$A^2(4 - A) = 8$$
$$A^3 - 4A^2 + 8 = 0$$

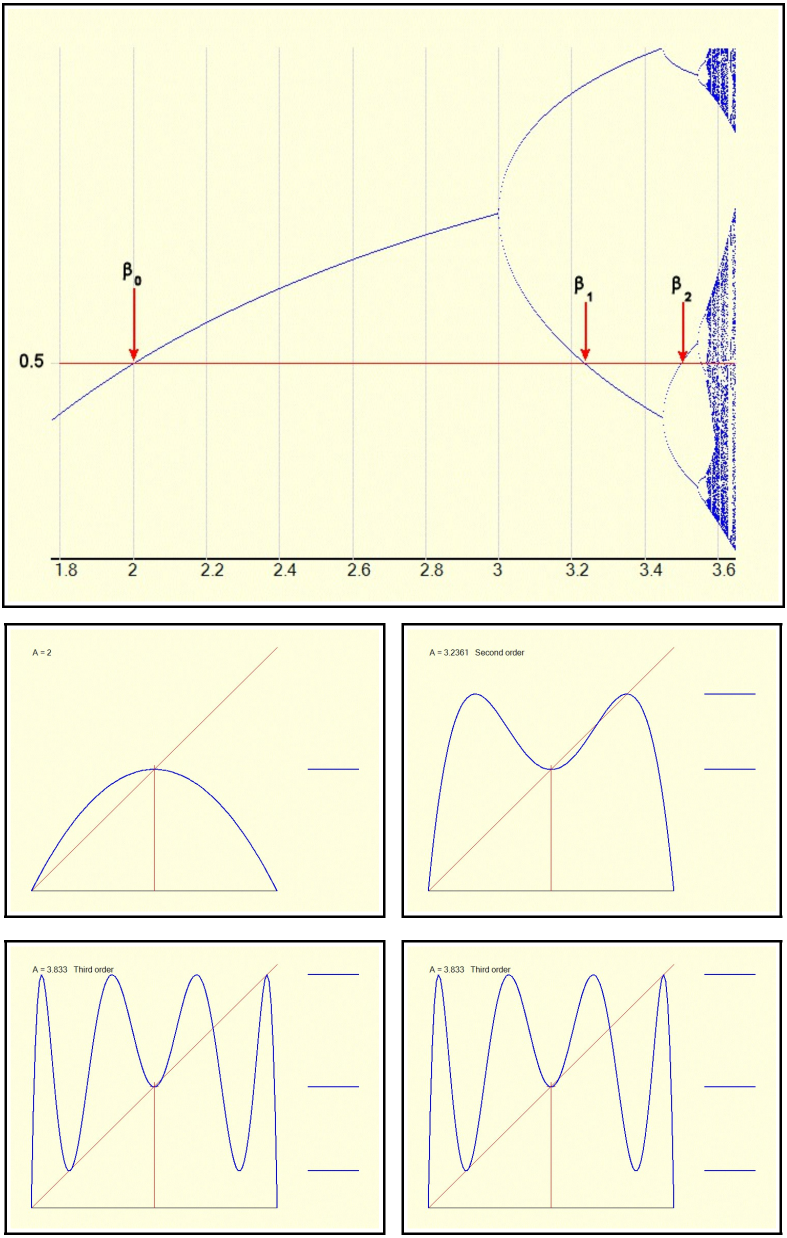
Although this is a cubic equation, we already know one of the answers – $A = 2$ so we can factorize it:

$$(A - 2)(A^2 - 2\mathrm{A} - 4) = 0$$

so the solutions are: $A = 2$, $1 + \sqrt{5}$ and $1 - \sqrt{5}$. The answer we are looking for is $\beta_1 = 1 + \sqrt{5} = 3.2361...$

Superstable points β_2, β_3, etc. inside the period 4, 8 and subsequent regions can be found by similar methods but the algebra gets pretty complicated. You can, however, see where the superstable points are on the chaos map because they are the points where one of the lines of stability crosses the critical value of x (= 0.5).

The lower illustrations show how the critical value $x_c = 0.5$ always homes straight in on the correct value when A is one of the superstable values because at these values, the n^{th} order graph passes exactly through the point (0.5, 0.5).



β_0
β_1
β_2
0.5
1.8
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
A = 2
A = 3.2361 Second order
A = 3.833 Third order
A = 3.833 Third order

THE ONSET OF CHAOS

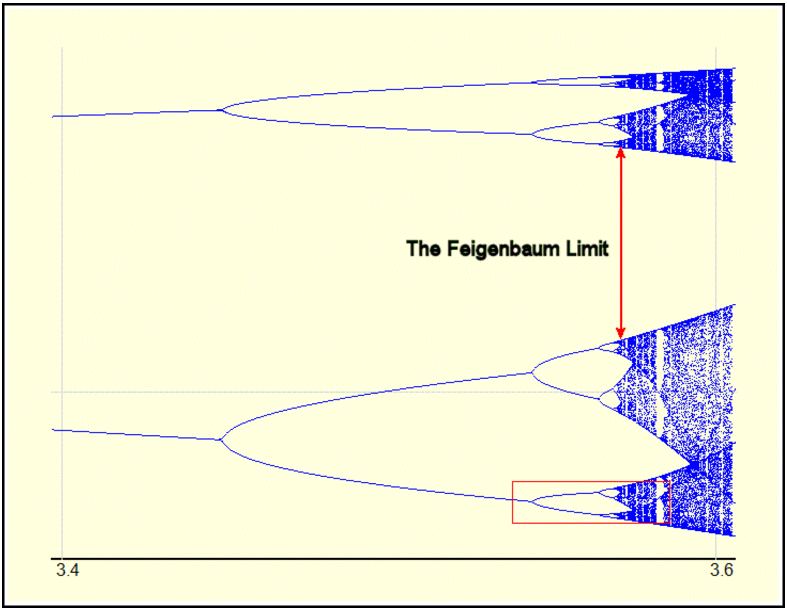
A = 3.56995...

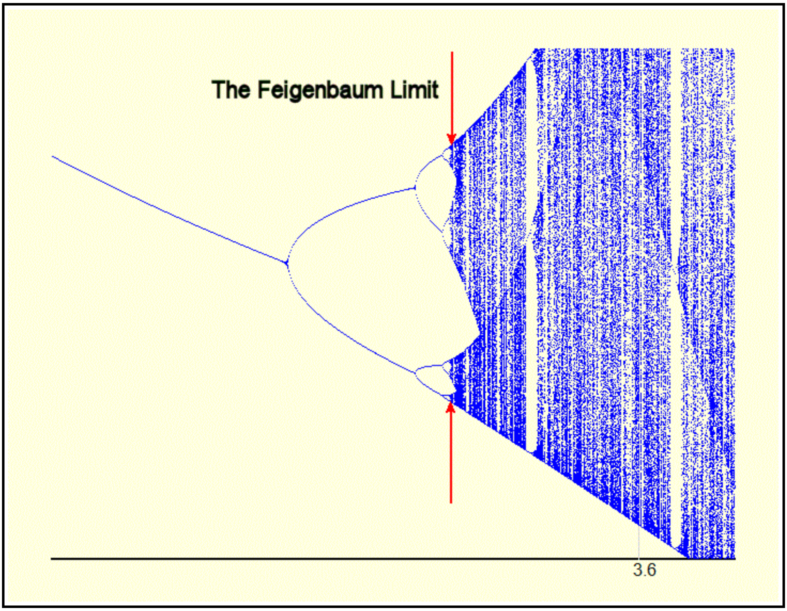
The process of bifurcation proceeds faster and faster as A increases and the interval between each successive bifurcation gets smaller and smaller in approximate geometric progression whose ratio approaches 4.6992... a number known as Feigenbaum's constant F_c[3]. This process reaches its limit when $A = \beta_\infty = 3.56995...$

Just below this number, x cycles through 2^n exact values where n is a finite number. Just beyond the Feigenbaum point, all these values have smeared out into a continuum which gradually merge into one another and eventually, when A is nearly equal to 4, spread right across the range from 0 to 1.

The upper illustration opposite shows the region between 3.4 and 3.6. The lower illustration shows the marked region. Only the first few bifurcations can be seen clearly even when magnified.

3 Technically F_c is equal to the limit of $\dfrac{\beta_{n+1} - \beta_n}{\beta_{n+2} - \beta_{n+1}}$ as n tends to infinity.


The Feigenbaum Limit
3.4
3.6


The Feigenbaum Limit
3.6

Islands Of Stability

α, β and γ points

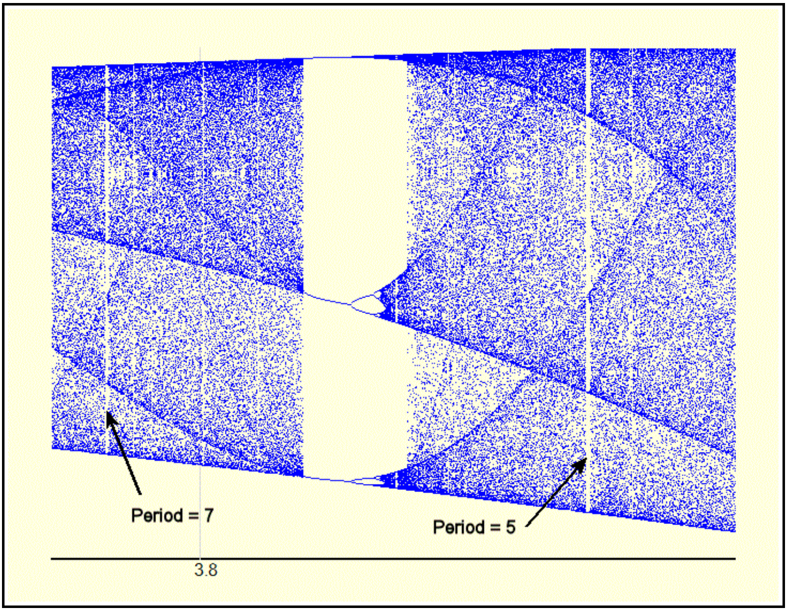
One of the most remarkable features inside the chaotic region is the appearance of clear bands the most prominent of which occurs where all the boundary lines touch in three places. This is illustrated opposite and it will be immediately obvious that it represents a range of values of A which has a period of 3, doubling to 6, 12, 24 etc. To the right of this band a much thinner band can be seen with a period of 5 and another on the left which has a period of 7. What causes these bands and how can we find other bands with specified periods?

As we saw on page 22, the superstable point β_0 inside an island of stability of period p can be found by solving the equation $f^p(A, 0.5) = 0.5$ but this is impossible to do algebraically in all but the simplest cases. Instead you can use a computer program such as Chaos Explorer to discover those values of A when the p^{th} order graph passes though the point (0.5, 0.5) The lower illustration shows the third order cobweb diagram with a value of $A = 3.8325$ which is the centre of this basin of attraction β_1.

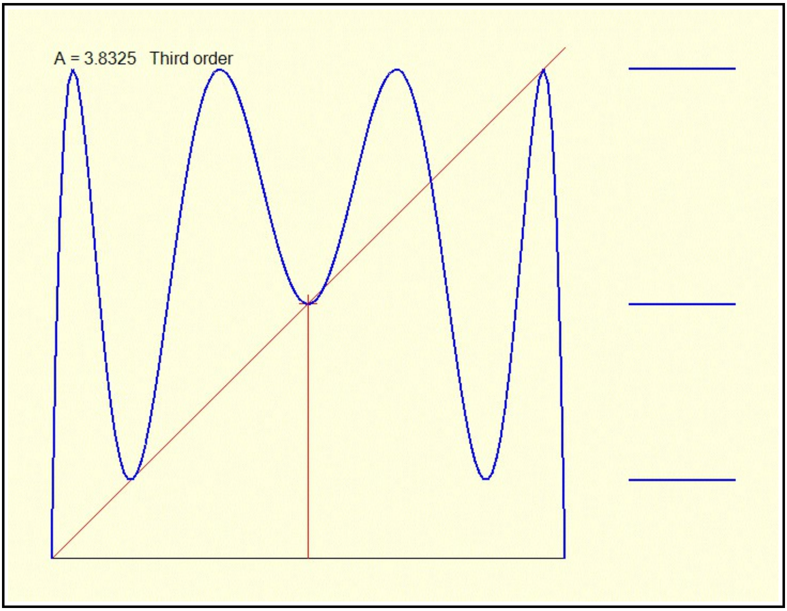
A vital point to notice here is that, as was proved on page 18, the gradient of the f^1 curve at this point is always zero. It follows that at nearby values of A, the gradient will always have a magnitude less than 1 and that the stable region will always have finite width.

The point where the stable region actually starts is the slightly smaller value of A when the central loop just *touches* the 45° line and has the approximate value of 3.82840. I call this the α point of the region. The point where the stable region ends is the γ point and is the limit of β_n as n tends to infinity. The α and β points are the roots of a rational polynomial and are therefore algebraic numbers. The γ point, however, is not and is probably always transcendental.

Since the α and β points are algebraic it follows that the number of basins of attraction, though infinite, is countable.



Period = 7
Period = 5
3.8



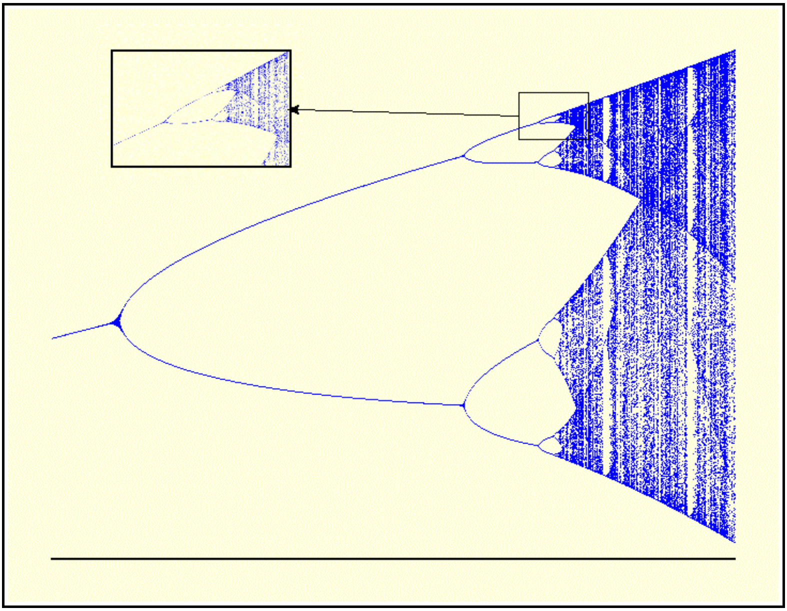
A = 3.8325   Third order

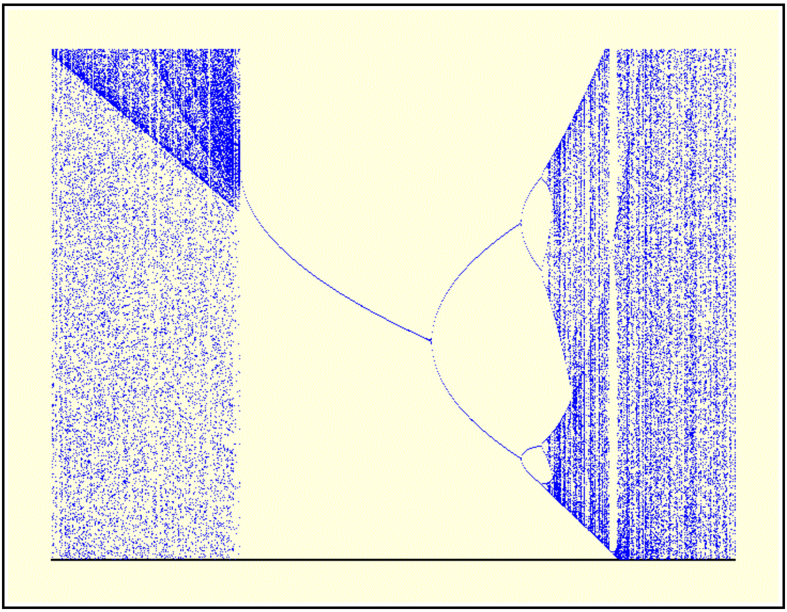
Fractals In Chaos

At a glance you might think that the top illustration opposite is just another picture of the chaos map. In fact it is the tiny region at the very top of the order 4 cycle and covers values of A from 3.54 to 3.58. The inset shows the equivalent region magnified again. It is clear that the map displays the self-similarity which is the hallmark of a fractal.

The lower illustration shows the region in the centre of the chaos diagram at $A = 3.8325$. Once again we see the characteristic pitchfork doubling. In fact you can find this pattern repeated over and over again within the chaotic region, wherever there are 'islands of stability'.

What is the reason for this fractal structure? On page 22 we saw that the first superstable point β_0 occurs because the $f^1(A, x)$ line passes through the point where $x = x_c = 0.5$. Here the gradient of the line is always zero. Now as A is increased, the gradient of the line where it crosses the 45° line increases and at the first bifurcation point it equals 1 (see page 10). We have also seen on page 18 that the second order f lines have a similar shape (only with more humps) and that they do exactly the same thing as A is increased. It is not surprising then that every superstable point in the middle of a basin of attraction bifurcates in exactly the same way that the first one does.





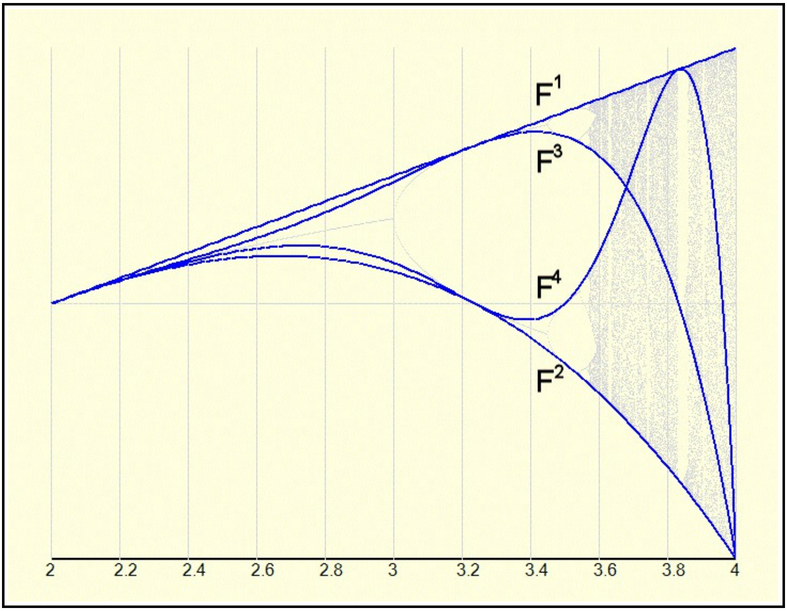
Boundary Lines

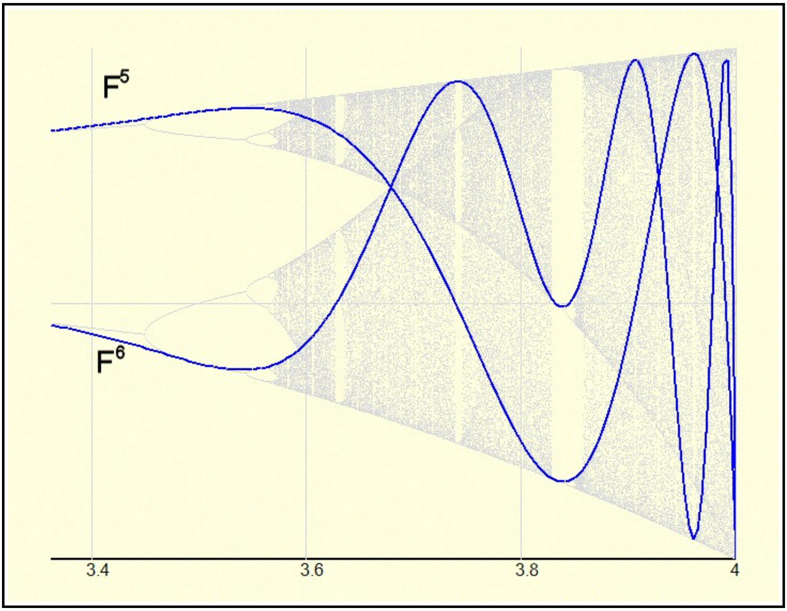
The next question to be addressed concerns the boundaries of the chaotic region in the vertical direction and what causes the differences in density of the points within it. To answer these questions it is useful to emphasise the first 4 iterations of x (starting with the critical value of $x_c = 0.5$). This is shown in the upper graph opposite. The topmost line is $f^1(\mathrm{A}, x_c)$ and is straight; the other line is $f^2(\mathrm{A}, x_c)$ We shall call these functions F^1 and F^2 from now on. (Note that F^n is a function of A only; $f^n(A, x)$ is a function of both A and x.). Here is a list of the first few boundary functions:

$$
\begin{aligned}
\mathrm{F}^1 &= A/4 \\
\mathrm{F}^2 &= A^2(4 - A)/16 \\
\mathrm{F}^3 &= A^3(4 - A)(16 - 4A^2 + A^3)/256 \\
\mathrm{F}^4 &= A^4(4 - A)(\text{ a polynomial of order 10 })/65536
\end{aligned}
$$

It is immediately obvious that the equation becomes extremely complicated after even a modest number of iterations. (In fact, the order of the resulting polynomial is equal to $2^n - 1$ where n is the number of iterations.) The lower illustration shows F^5 and F^6. F^5 has one minimum and one maximum in the region $A > 3.6$ while F^6 has two minima and three maxima in this region. The first major crossing point is where the lines F^3 and F^4 cross. This is the first Misiurewicz point and will be discussed later (see page 36). In fact all the boundary lines pass through this point.

Since the logistic function is a maximum at the critical point, it is clearly not possible for x to exceed F^1, nor is it possible for it to go below F^2. This is why these lines form the boundaries within which x is constrained. The other lines appear as denser regions in the overall map because, on one side certain higher order lines are 'turned around' at this point just as, for example, the F^6 line is 'turned around' whenever it approaches the F^3 line in the middle of the period 3 basin.


F1
F3
F4
F2
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4

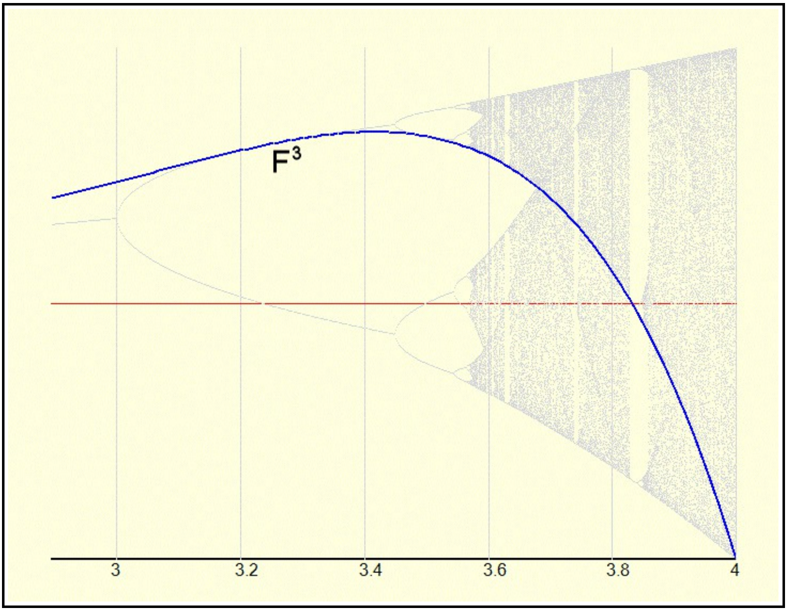

F5
F6
3.4
3.6
3.8
4

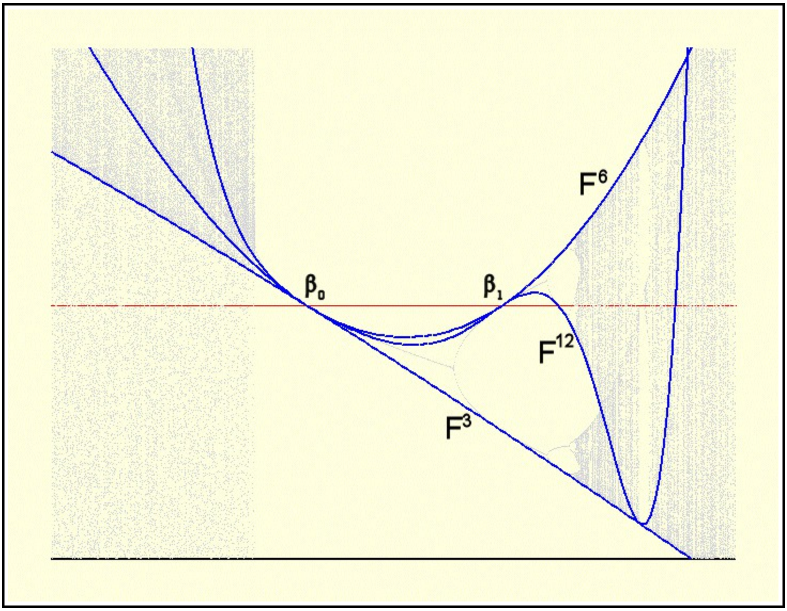
FINDING β POINTS

On page 22 it was stated that islands of stability occur when one of the functions $f^p(\mathrm{A}, x_c)$ passes through the point (0.5, 0.5). We can now express this idea more simply as $\mathrm{F}^p = 0.5$. In other words, basins of attraction occur whenever one of the boundary lines crosses the $x = 0.5$ line. This is shown in the upper illustration opposite. The F^3 line crosses the red line right in the middle of the order 3 basin of attraction.

We can also use subsequent boundary lines to find all the β points within a basin of attraction. The magnified illustration at the bottom shows the F^3, F^6 and F^{12} lines passing through the central 'bridge' of the basin. The F^3 line crosses the centre line once only but the F^6 and F^{12} lines cross it twice and thrice respectively. These crossing points are β_1 and β_2.

To find a basin of attraction of any period p, all you have to do is look at the line F^p and see where it crosses the centre line. You can immediately see that there will be only one basin of order 3.


F3
3
3.2
3.4
3.6
3.8
4


F6
β0
β1
F12
F3

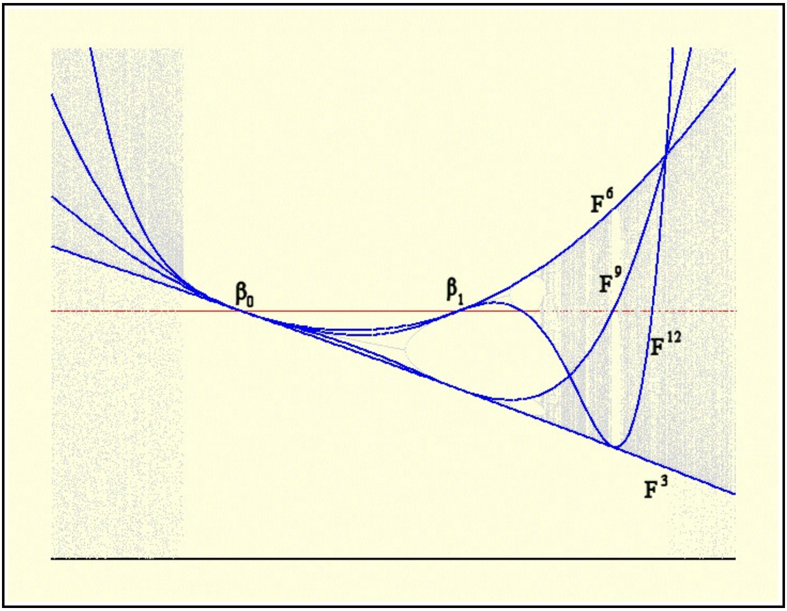
The F-Line Theorem

Now it may seem remarkable that, with all those wiggles, *all* higher order F lines can pass through the order 3 basin of attraction in just three places. How do they do it?

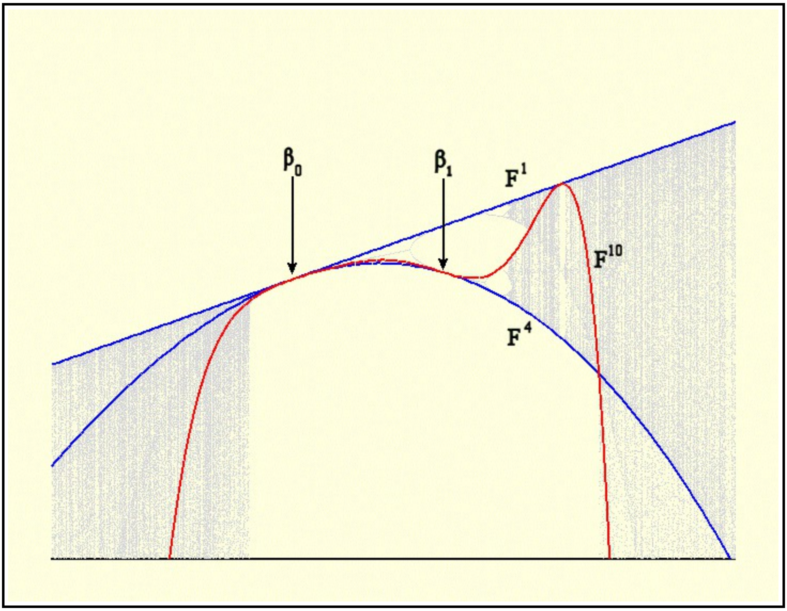
Suppose that the boundary line F^p crosses the centre line at $A = \beta_0^{p.}$ What this is saying is that starting with $x = 0.5$, after p iterations we return to 0.5. Now if we iterate for a further p iterations we will cycle through exactly the same numbers. It follows that if $F^p = 0.5$, then $F^{p+n} = F^n$ for all n. and that *all the higher order lines must pass through the same p points.*

But that is not all. A detailed look at the region where F^3 crosses the centre line reveals that not only do lines F^6, F^9 and F^{12} pass through the same point, they also *have the same gradient* at this point too. In fact it is general rule that whenever $F^p = 0.5$, then the *gradient* of F^{n+p} will be equal to the *gradient* of F^n for all n as well. (You can see this at the point β_1 where the F^6 line crosses the centre line. Here F^3 has the same gradient as F^9 and F^6 has the same gradient as F^{12}.) In fact, I would venture to suggest that at $A = \beta_0^p$ the k^{th} differential of the line F^{kp} is equal to the k^{th} differential of the line F^p but I do not have a proof of this. The reason for this is simple. Since all the values of A near β_0 are stable, the higher the order of the line F^{kp} the closer it will lie to the line F^p.

The upshot of all this is that the higher order lines cannot just wiggle where they please. The lower illustration shows the F^{10} line (in red) at the top of the period 3 island. Because of the F-line theorem, it has to have the same gradient as $F^1$and F^4 (and F^7) at the β_0 point and the same gradient as the F^4 line at the β_1 point. This effectively constrains it to lie between the boundary lines F^1 and F^4 across the whole island of stability. It also means that within an island of stability of period p, all the F lines of higher order will exhibit an effective maximum or minimum. The converse is also true. If you see a place where F^n has a maximum or a minimum, there must be an island there of period less than n.



β_0
β_1
F^6
F^9
F^{12}
F^3



β_0
β_1
F^1
F^{10}
F^4

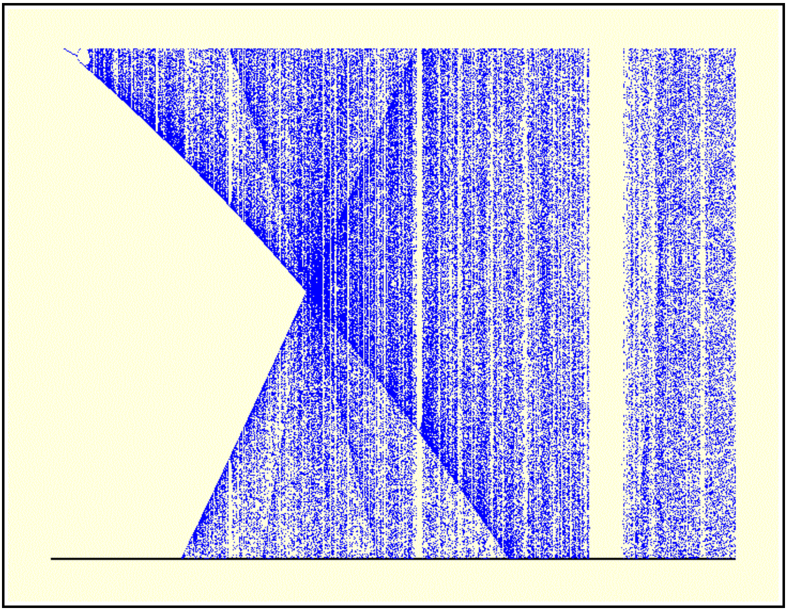
PRE-PERIODIC POINTS

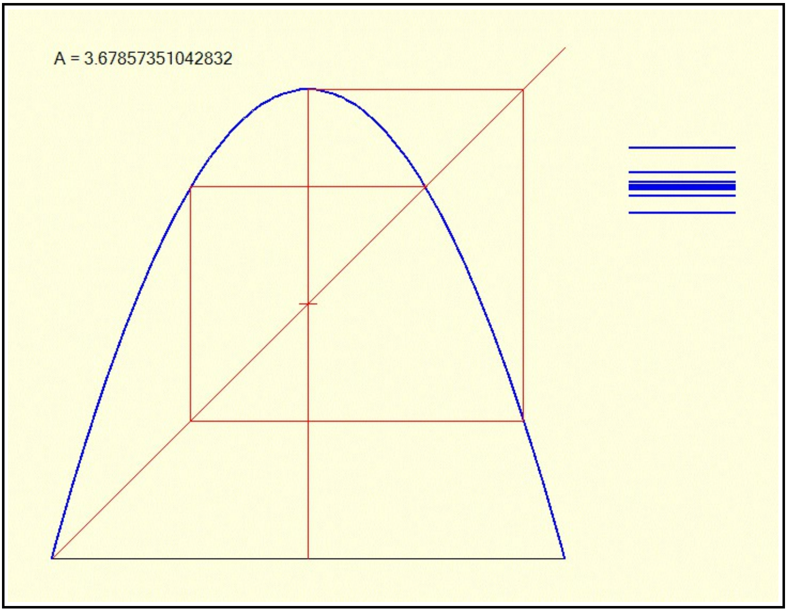
$A = 3.67857351042832...$

In addition to the 'islands of stability' discussed above, there are other cyclic points called Misiurewicz points. Instead of homing in on a stable cycle these values of A cause an initial value of $x = x_c$ to bounce around for a while (a phase called the pre-period) and then enter a periodic cycle. If after a pre-period of n iterations the point enters a cycle of period p then this value of A is designated $\mathbf{M}(n, p)$. What this means is that the $(n + p)^{\text{th}}$ iterate of the critical point must equal the n^{th} iterate i.e. $F^{(n+p)} = F^n$.

Now it was noted on page 22 that the major crossing point (illustrated opposite) is the place where lines F^3 and F^4 meet. This means that this is, in fact, $\mathbf{M}(3,1)$ and it has the approximate value of 3.67857351042832... The cobweb diagram shows how x jumps from 0.5 to 0.92, then to 0.27 from which point it happens to hit the place (0.728) where the parabola crosses the 45° line thus entering a cycle of period 1. The problem is – this point is a repellor (because the gradient of the parabola at this point is greater than 1) so if A is not *exactly* equal to the right number, x will gradually diverge away from its cyclic value. This is indicated by the horizontal lines on the right which show the next dozen or so iterations.

We have seen that, as n increases, the boundary line F^n acquires more and more wiggles and the number of places where it can potentially cross one of the lower F lines increases enormously. There are therefore an awful lot of Misiurewicz points so it is not inconceivable that *all* values of A lead eventually to a periodic pattern. But do they? We shall return to this question again and again.




A = 3.67857351042832

Initial Sensitivity

One of the characteristics of chaos is what is called its 'sensitivity to initial conditions'. What this means is that, if A is a non-periodic (chaotic) value, then if you iterate two very slightly different values of x, the difference will increase exponentially. This behaviour is illustrated in the diagram opposite. Two very similar initial values (0.100 and 0.101) are iterated for 4 different values of A. When $A = 2$ the function homes in on a single stable value. With $A = 3.2$ the function bifurcates. At $A = 3.5$ the function has bifurcated again but the two starting values remain in step. By the time we get to $A = 3.7$, though, differences appear after 4 or 5 iterations which soon magnify until there is no apparent correlation between the two graphs at all.

We can get a handle on the speed of this divergence by noting that when two different values of x separated by a small quantity δx are iterated, the difference in the values after iteration δy will be equal to $G(x)\delta x$ where $G(x)$ is the gradient of the function at x. After n iterations the difference will be approximately

$$\delta y \approx G(x_0).G(x_1).G(x_2)...G(x_n)\,\delta x$$

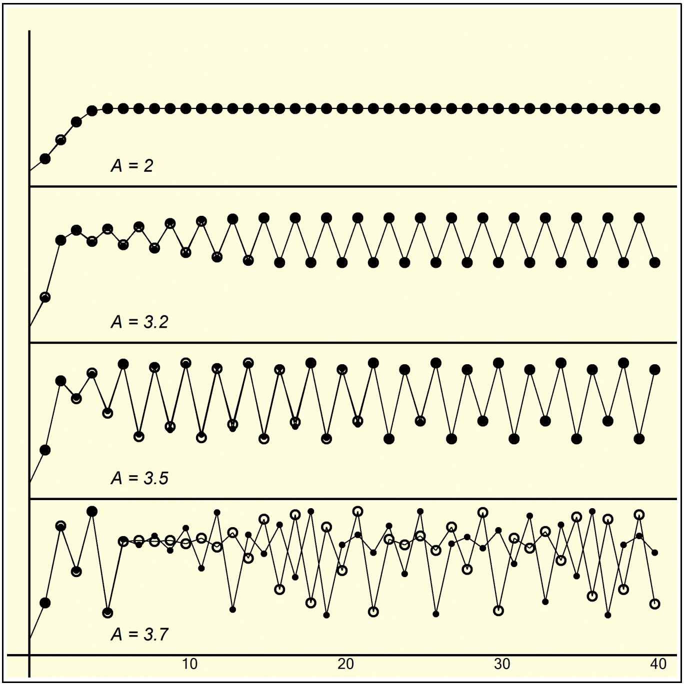
(provided that δy remains small). Alternatively we can say that δy will increase exponentially, multiplying by a factor G at every iteration where G is the *average gradient* (technically the geometric mean gradient[4]) which x encounters. i.e.

$$\delta y \approx G^n\,\delta x$$

where $$G = \sqrt[n]{G(x_0).G(x_1).G(x_2)...G(x_n)}$$

We have already seen that when the average gradient is less than 1, the function is stable; but if the average gradient is greater than 1 the system will exhibit extreme sensitivity to its initial conditions.

4 Since we are not interested in the sign of the difference, we use the absolute value of the gradient.



A = 2
A = 3.2
A = 3.5
A = 3.7
10
20
30
40

LYAPUNOV EXPONENT

The average (or geometric mean) gradient G of an orbit is a measure of its stability. Since the gradients along the way must all be multiplied together, it is more convenient to add their logarithms and take the (regular) mean of these. The result is called the Lyapunov exponent of that point.[5]

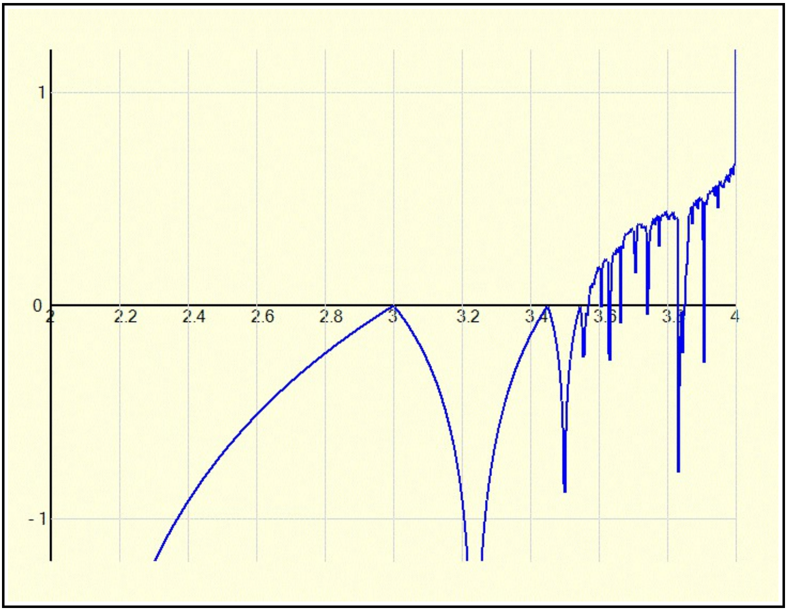
The upper graph opposite shows the Lyapunov exponent for the logistic function between 2 and 4 averaged over 100 iterations after the first 1000. Negative values indicate values of A which converge on a stable value (superstable points have a Lyapunov exponent of $-\infty$). Where the graph is zero, this is a point of neutral stability e.g. a place where the chaos diagram bifurcates. Where the graph is positive, the differences increase exponentially which indicates that these points are unstable. Such points include the Misiurewicz points and other totally chaotic points (if they exist).

The lower graph shows the expanded region close to the period 3 island. In addition to the large island of stability you can see that several more spikes have appeared which dip below the axis and which represent much smaller islands of stability. In fact, however much you magnify the graph, more and more negative spikes appear and the only reason that we don't see them is that they are too thin and fall between the calculated values.

In fact, even those points which remain above the axis cannot be guaranteed to be aperiodic because they might be Misiurewicz points

So the question still remains – are there *any* genuinely chaotic points at all?

5 Strictly speaking, it is not possible to talk about the Lyapunov exponent *of a single point* because what is being measured is the rate at which *two nearby points* diverge. If you use the algorithm to work out the Lyapunov exponent for a Misiurewicz point you will get a false answer because the gradients will repeat over and over again. In practice this does not matter because Misiurewicz points are relatively rare and the computer will almost always calculate the exponent of a nearby (aperiodic) point.


1
0
-1
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4


1
0
-1
3.8

Period And Pre-Period

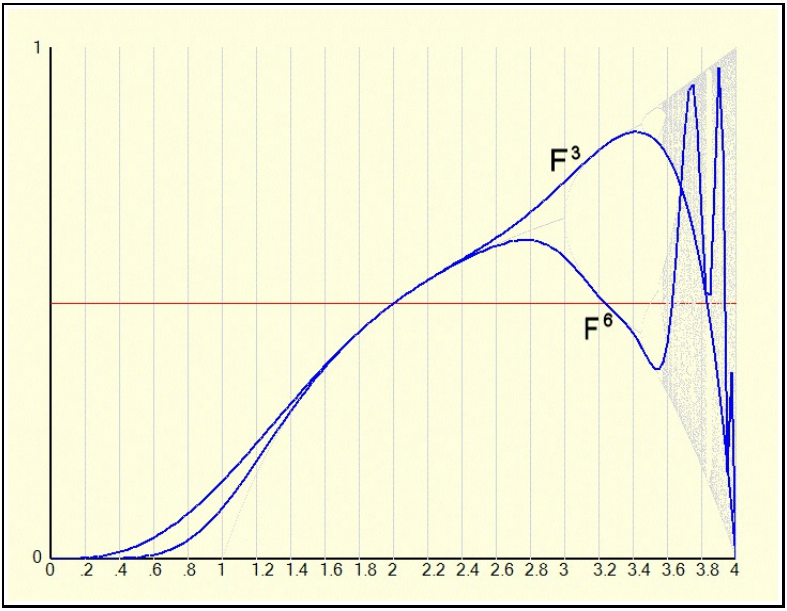
Consider the points where F^n crosses F^m (where $m > n$). These values of A will be the solutions to the equation $F^m = F^n$ and since the equation F^m has order 2^{m-1} there will, in general, be 2^{m-1} solutions. All of these solutions will have a pre-period of n and an eventual periodicity of $m - n$ because, having reached a certain value x after n iterations, it must return to x after a further $m - n$ iterations.

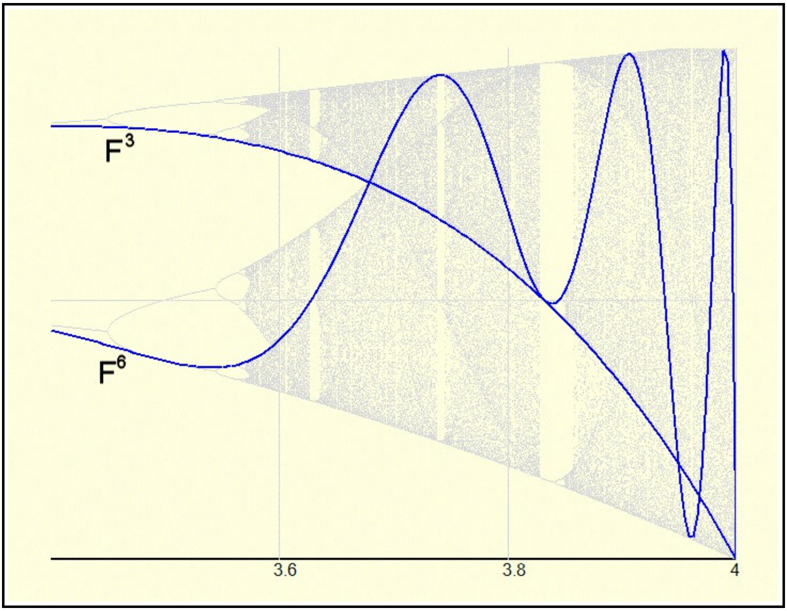
Take the case of $n = 3$ and $m = 6$ illustrated opposite. There will be 63 solutions but many of them are either degenerate or imaginary. Some of the real solutions are accounted for by the solution $A = 0$. Many more occur at $A = 2$ where both lines touch at the centre of the order 1 basin of attraction. We expect all the real solutions to have an eventual periodicity of 3.

The lines cross at the first Misiurewicz point **M**(3, 1) which has a pre-period of 3 and a periodicity of 1. (Any point with a periodicity of 1 also has a periodicity of 3.) At $A = 3.83$ the lines just touch. This is the centre of the basin of attraction of period 3 and can be seen more clearly in the second illustration. They cross twice more at two further Misiurewicz points, both of them designated **M**(3, 3).

We can draw some general conclusions straight away. Where there are degenerate solutions (i.e. where the F lines touch), these are the centres of basins of attraction; where there are unique solutions (i.e. where the lines cross) there are Misiurewicz points.

Just as there are a countable number of basins of attraction (see page 24) there must also be a countable number of Misiurewicz points.


1
0
F3
F6
0
.2
.4
.6
.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4


F3
F6
3.6
3.8
4

Misiurewicz Points

Below is a list of Misiurewicz points generated by the boundary lines 3 through 7.

A	**M** *point*	*Crossings*
3.59257	**M**(5, 2)	5×7, 6×8
3.67850z	**M**(3, 1)	3×4×5×6×7
3.76489	**M**(4, 2)	3×5, 4×6
3.78	**M**(3, 4)	3×7
3.79113	M(6, 1)	6×7
3.87655	**M**(6, 1)	6×7
3.89	**M**(3, 2)	3×5
3.92775	**M**(4,1)	4×5×6×7
3.94282	**M**(5, 2)	5×7
3.95	**M**(3, 3)	3×6, 4×7
3.97	**M**(3, 3)	3×6, 4×7
3.97459	**M**(4,2)	4×6, 5×7
3.98257	**M**(5, 1)	5×6×7
3.98734	**M**(4,3)	4×7
3.98910	**M**(3,4)z	3×7
3.99125	**M**(3,4)z	3×7
3.99228	**M**(4,3)	4×7
3.99378	**M**(5, 2)	5×7
3.99570	**M**(6, 1)	6×7

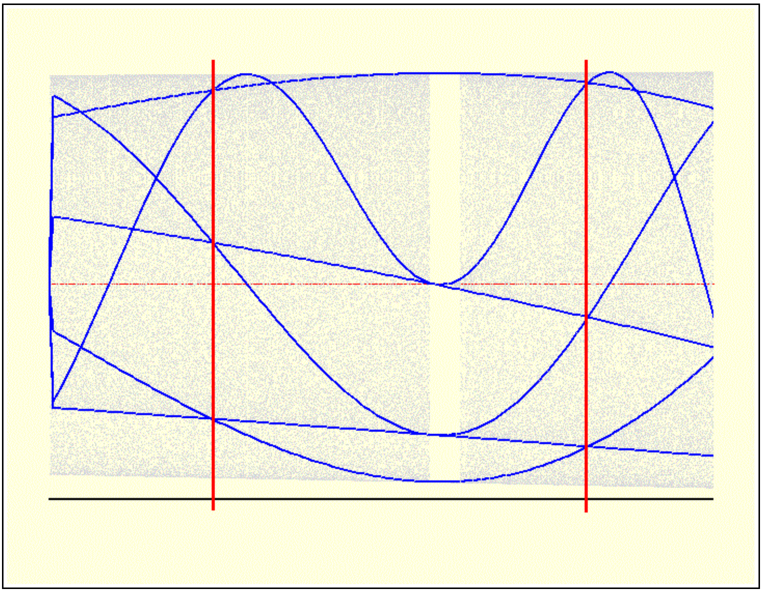
The points $A = 0$ and $A = 4$ are Misiurewicz points **M**(0, 1) and **M**(2, 1)

The designation **M**(n, p) is not unique. There may be more than one Misiurewicz points with the same pre-period and period. For example, there are two **M**(3, 3) and three **M**(6, 1) in the list.

Any Misiurewicz point with designation **M**(n, p) can also be designated **M**($n + i$, jp) where i and j are integers. For example **M**(3, 4) could equally well be called **M**(5, 12) because once the point has completed its genuine pre-period, all subsequent points are periodic; and anything with a period of 4 also has a period of any multiple of 4. The consequence of this is that, just knowing the designation does not necessarily tell you the pre-period and the period unless you also know that the designation has been reduced to its lowest terms. Nor is it possible to reduce a designation without further information.

If you know the pre-period n and the period p, you can deduce that the lines F^n and F^{n+p} cross at that point. You can also deduce that the lines F^{n+i} and F^{n+p+i} also cross at that point (where $i < p$) but at a different value of x. So at **M**(3, 3), line F^3 crosses F^6, line F^4 crosses F^7, and line F^5 crosses F^8 in three different places.

The following illustration shows the region between 3.94 and 3.98. The two Misiurewicz points **M**(3, 3) are indicated in red. All higher order F lines must pass through these three crossing points as well.



CRISIS POINTS

On page 12 it was noted that the point where F^3 crosses F^4 (**M**(3, 1)) is associated with the first bifurcation point; likewise the second bifurcation point is associated with another Misiurewicz point – **M**(5, 2) where F^5 crosses F^7 and F^6 crosses F^8. (It is no accident that the former has period 1 while the latter period 2.)

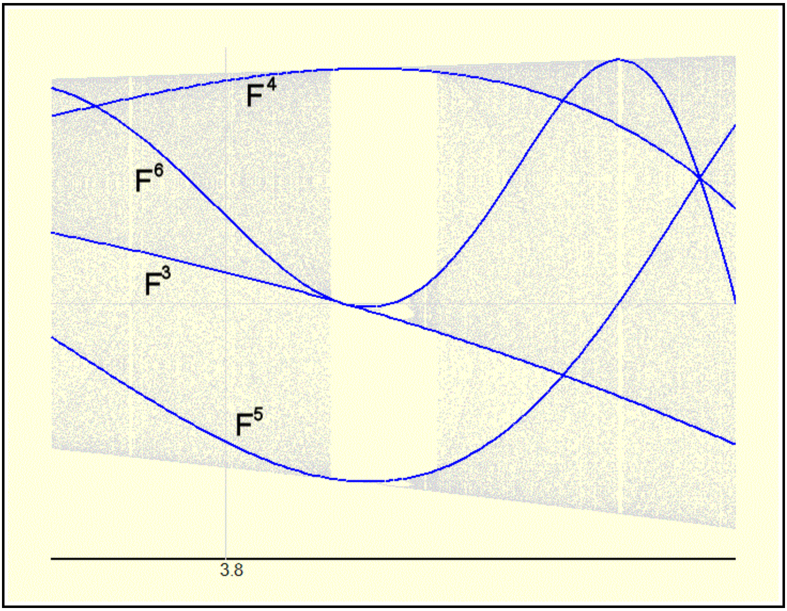
In fact there is a whole sequence of Misiurewicz points which mirror the bifurcation points in reverse, as it were. The next one in the sequence will be the crossing point of the F lines 9 & 13, 10 & 14, 11 & 15, 12 & 16 etc. etc. In fact, every basin of attraction ends with an infinite sequence of period doublings, each of which has its own unique associated Misiurewicz point. (This would appear to suggest that there are infinitely more Misiurewicz points than basins of attraction. This would be a mistake though as $\infty \times \infty$ is still ∞.)

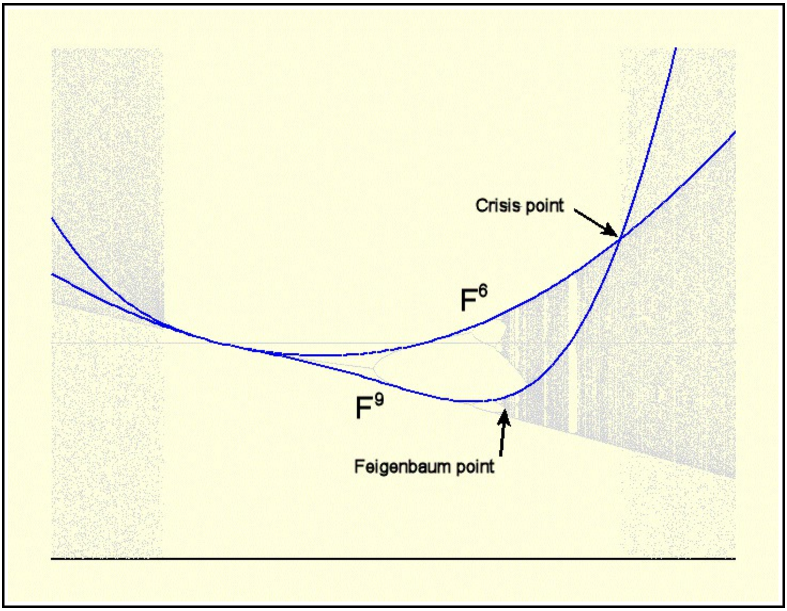
But not all Misiurewicz points are associated with a period doubling point. The illustration opposite (top) shows the period 3 band and boundary lines F^3, F^4, F^5 and F^6. The principal Misiurewicz point associated with the period doubling sequence which ends this region of stability is clearly shown where F^3 crosses F^5 (lower) and F^4 crosses F^6 (upper). But unlike the case of the main period doubling sequence, the band of stability comes to an end long before this value is reached.

An enlargement of the central 'bridge' showing the boundary lines F^6 and F^9 shows why. The F^9 line, once released from the constraints imposed on it by the band of stability, shoots off and crosses the F^6 line and enters what was previously regarded as a 'forbidden' region. Higher order lines do so with even more enthusiasm.

This point is known as a 'crisis' point. In general, the central 'bridge' of an island of stability of period *p* will be bounded by the F lines *p* and 2*p*. The crisis point will be where 2*p* F line is crossed by the 3*p* line.

(Note that the island of stability ends at the γ or Feigenbaum point which is well before the crisis point.)


F4
F6
F3
F5
3.8


Crisis point
F6
F9
Feigenbaum point

M-Points and Basins

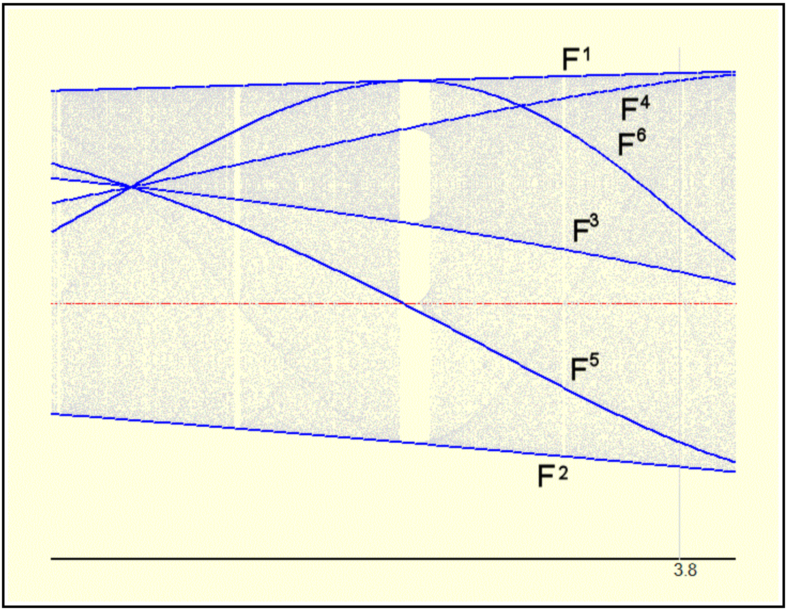
We have seen that every Misiurewicz point is the crossing point of two boundary lines (see page 42). We have also noted that the β points (the centres of the basins of attraction) are places where all the groups of boundary lines above a certain value have equal gradients (see page 34). It is important to prove that none of the Misiurewicz points can lie inside a basin of attraction. Take, for example the period 5 basin of attraction at $A = 3.74$. This is the point where F^1 to F^5 all have different gradients and cross the region in different places but F^6 just touches the line F^1. We can see straight away that the former lines form the 'bridges' across the window and that this is the reason that this window has period 5 but can we prove that this will always happen? Isn't it possible that there might be a case where two of the lower order lines actually cross inside the window?

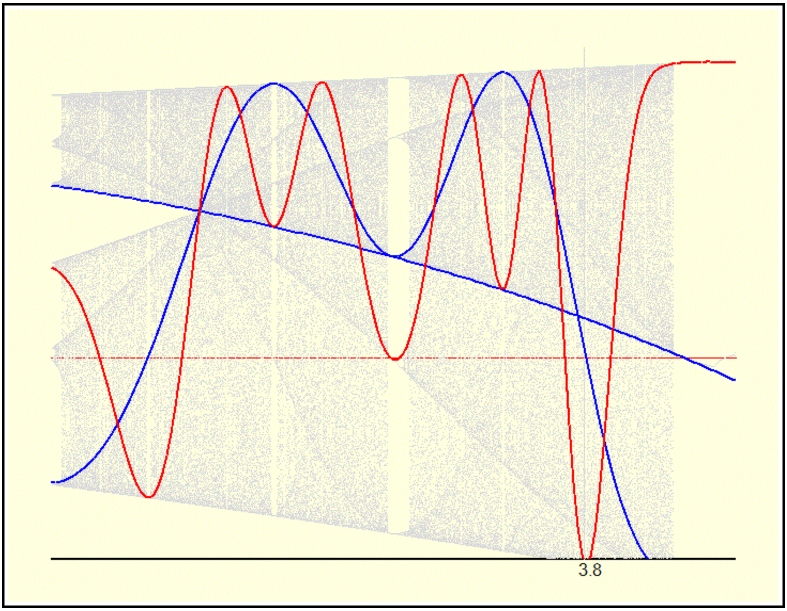
On page 32 we showed that whenever any boundary line F^n passes through the critical point $x = 0.5$ there will be a basin of attraction there of period n.

Now if two of the lower order F lines F^p and F^q were to cross or touch at some other value of x inside this basin, then it would have a smaller period because $p - q$ must be less than n.

For example, in the lower illustration opposite F^{10} shown in red just touches the $x = 0.5$ line. This basin of attraction therefore has period 10. We also note that the two boundary lines F^3 and F^8 touch inside this basin which means that the basin also has period $8 - 3 = 5$.

It is, of course, the period 5 basin. True period 10 basins will only be found where the F^{10} line *crosses* rather than *touches* the $x = 0.5$ line and in these basins, none of the F lines 1 – 9 will cross.


F¹
F⁴
F⁶
F³
F⁵
F²
3.8

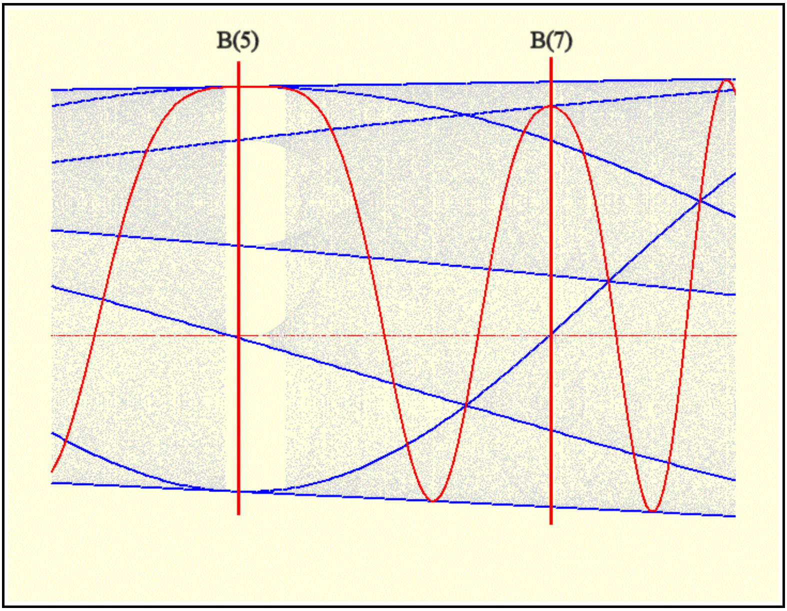

3.8

THE M-THEOREM

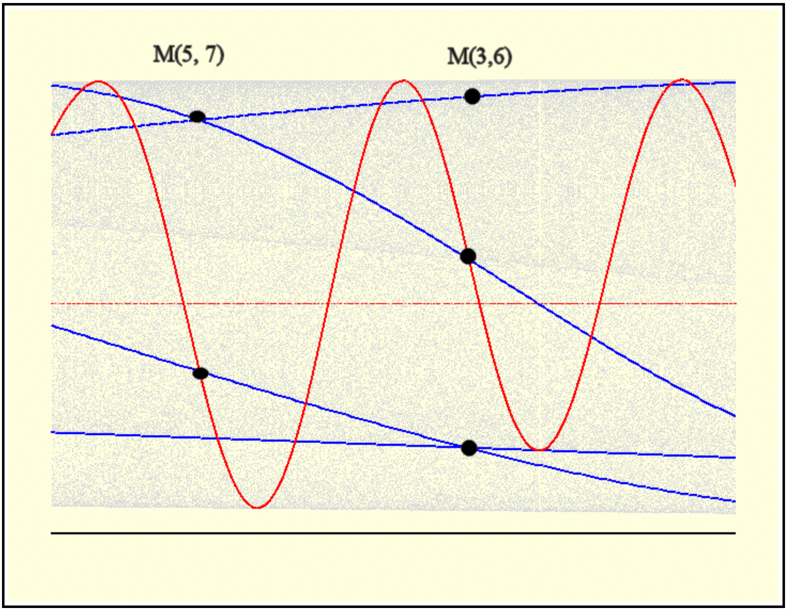
We now come to the most important theorem of them all. Not only is it true that all Misiurewicz points lie outside the basins of attraction, it is also the case that *between any two Misiurewicz points there are always an infinite number of basins of attraction* and that *between any two basins of attraction there are always an infinite number of Misiurewicz points*.

The reasons for this are pretty obvious. Consider the two basins of attraction shown in the upper illustration opposite which have period 5 and 7 respectively. On page 34 we noted that every F line has to pass through the same 5 or 7 bridges across these basins. In between they must reorganise themselves to get in the right order and inevitably, some will cross. Indeed the higher order lines (for example the line F^{11} shown in red) will cross the lower order line many many times. All these crossing points are, of course, Misiurewicz points.

The lower illustration shows the region between the two Misiurewicz points **M**(5, 7) (which has a period of 2) and **M**(3, 6) (which has a period of 3). The points in the periodic cycles are indicated with black dots. Now it was pointed out on page 45 that all higher order lines must pass through one or other of these points and in order to get from one set of points to the next, many of these lines will cross the $x = 0.5$ line many times. (For example the F^{10} line shown in red.) At all these points there will be basins of attraction.



B(5)
B(7)



M(5, 7)
M(3,6)

Aperiodic Points

We have seen that every Misiurewicz point is the crossing point of two of the F functions and is therefore the root of a polynomial of order $2^{(n+p)}$ where *n* is the pre-period and *p* the period. We also know that Misiurewicz points are *never* found inside basins of attraction This is because inside a basin of attraction the point x_c homes in on the attractor - it doesn't become periodic. (The exception is, of course, the attractor itself which could be regarded as a Misiurewicz point with a pre-period of zero.) So in spite of there being an infinite number of basins of attraction, each of finite size, there are still numbers left over.

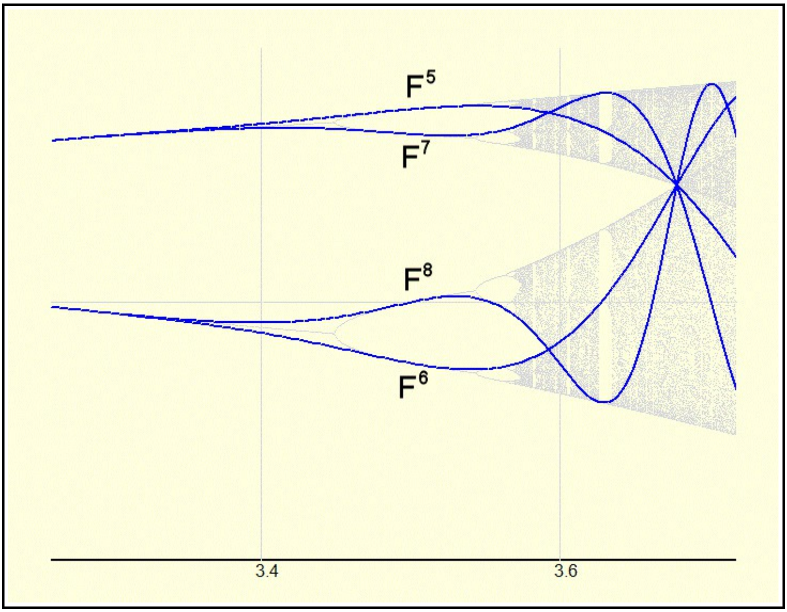
Now we know that there are countless numbers (literally an uncountable number) of transcendental numbers which are not the root of any polynomial. π is a good example. So the point $A = \pi$ is not a Misiurewicz point. It could, however be a point inside one of the basins of attraction. (It is, of course, inside the basin of attraction of period 2.) The trouble is, there is no general algorithm for determining whether a given value of *A* is inside one of the basins of attraction or not. In fact, since the basins of attraction all have finite width, any random number is far more likely to be in one of these basins than not so this leaves us in the following impossible position of suspecting that aperiodic points exist in uncountable numbers but not being able to name a single one of them precisely!

One potential candidate for an aperiodic point is the limiting value at which the initial period doubling sequence tends to infinity – the Feigenbaum points. Here *A* is approximately equal to 3.56995...

Another way to approach this point is to consider the sequence of principal Misiurewicz points discussed on page 46. The principal one is found where F^3 crosses F^4. The next most important one is found where F^5 crosses F^7 and F^6 crosses F^8. These are shown in the diagram opposite. The next one is at the crossing point of F^{12} and F^{16}. It should be clear that the limit we are seeking is one of the solutions to the equation $F^a = F^b$ where $a = 3\times2^n$ and $b = 4\times2^n$ as *n* tends to ∞. This point could be regarded as a Misiurewicz point whose pre-period is infinite.

Since every basin of attraction ends with a period doubling sequence, this type of point is very numerous but they are none the less countable because, owing to the fact that the β point is algebraic (see page 26), there are a countable infinity of basins of attraction.

So the question of how many aperiodic points there are remains unresolved. Are all the transcendental numbers which are not Feigenbaum points inside basins of attraction? Or is there an uncountable infinity of such numbers outside the basins of attraction? I do not know.



THE MANDEBROT AXIS

The Mandelbrot map is generated by iterating the function $\boldsymbol{z}' = \boldsymbol{z}^2 + \boldsymbol{C}$. where $\boldsymbol{z}$ and $\boldsymbol{C}$ are complex numbers starting from the critical point $\boldsymbol{z_c} = 0$. Now if $\boldsymbol{C}$ is a real number i.e. its imaginary component is zero) then $\boldsymbol{z}$ will also be just a real number and the whole process reduces to iterating the function $x' = x^2 + C$.

This function is an inverted parabola and its chaos map is shown opposite. It is immediately apparent that it shows exactly the same characteristics as the Logistic map but in reverse and the relevant range is 0.25 down to -2.

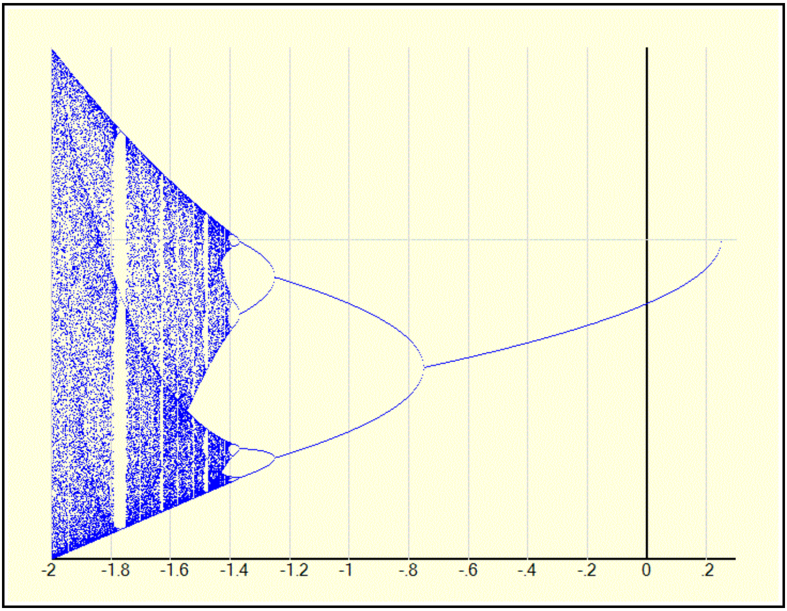
There is, in fact a simple relation between the Mandelbrot constant C and the logistic constant A so if you know where a certain feature like a bifurcation or a basin of attraction occurs in one map, you can find it in the other. The relation is:

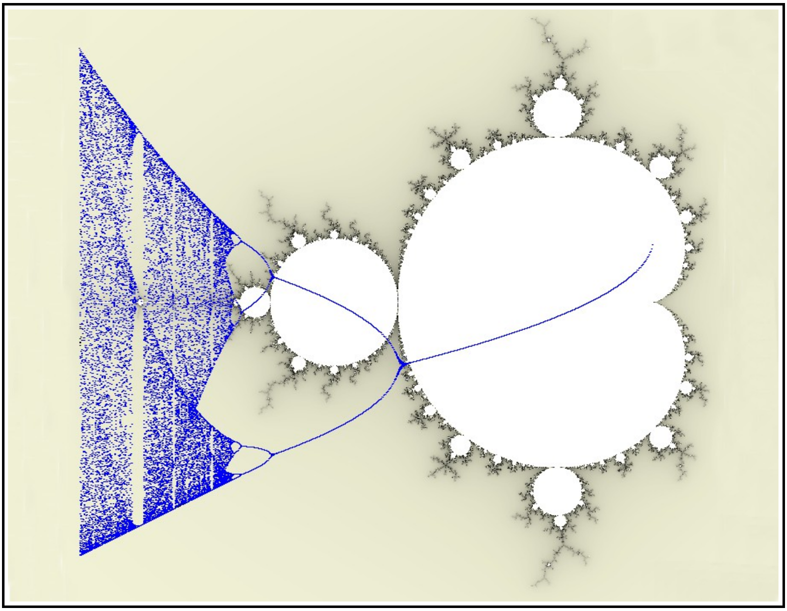
$$A = 1 + \sqrt{1 - 4C} \text{ or } C = A(2 - A)/4$$

The diagram below shows this map superimposed on the Mandelbrot map and the relation between them is immediately apparent.

The first bifurcation occurs at the place where lobe 2 meets the main cardioid and the successive bifurcations at the junctions of lobes 2\2, 2\2\2, 2\2\2 etc. (For an explanation of the lobe labels, see my book on the Mandelbrot Map.)

But the big revelation comes when you look at where the period 3 basin occurs. It is exactly where the large minibrot occurs on the antenna. In fact, every basin of attraction in the logistic map corresponds to a minibrot in the Mandelbrot Map!


-2
-1.8
-1.6
-1.4
-1.2
-1
-.8
-.6
-.4
-.2
0
.2

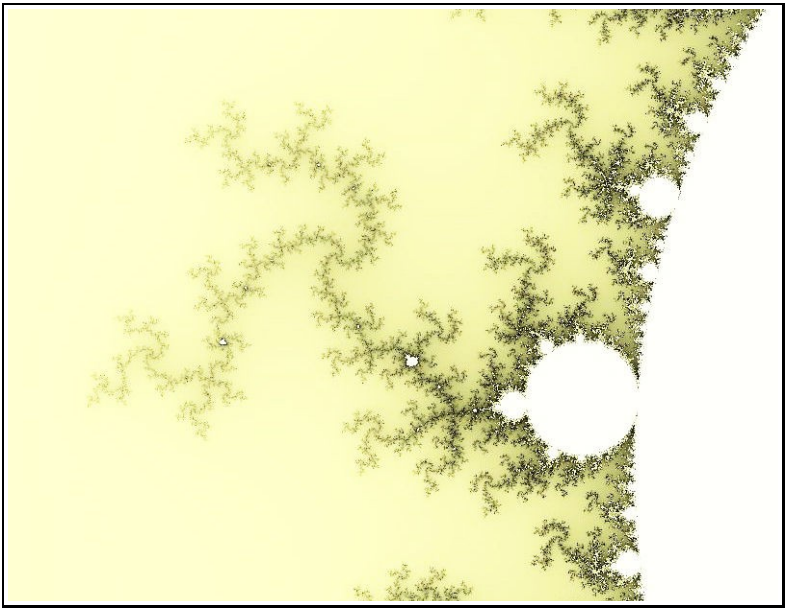


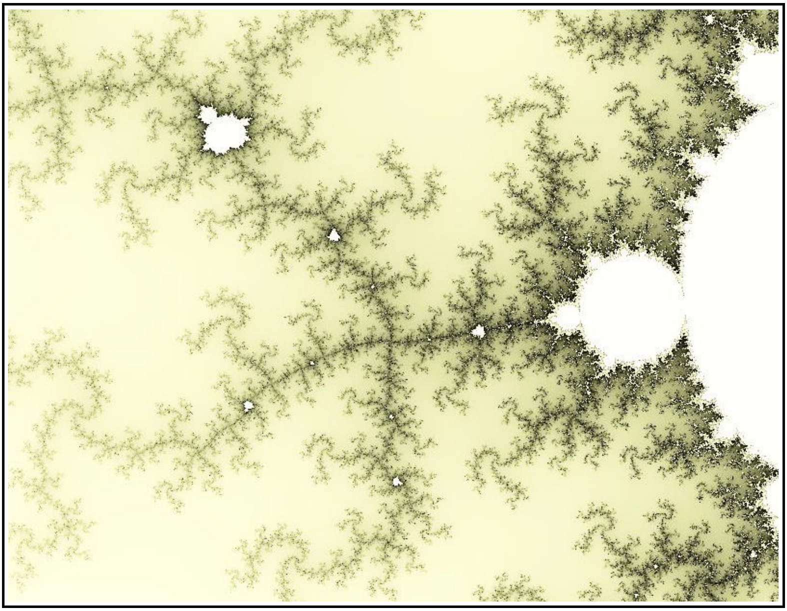
MINIBROTS AND BASINS

We have noted that the α point of a basin of attraction is the root of a finite polynomial and is therefore an algebraic number. This means that there are a countable number of basins of attraction and a countable number of minibrots along the axis of the Mandelbrot map. In between these minibrots are the Misiurewicz points. These points (which I call synapses) can sometimes be recognised in the Mandelbrot Map because they are places where the filaments divide into multiple branches. For example, the filament which attaches itself to the lobe at the top of the map soon splits into 2. (This synapse is said to have order 3 because 3 branches meet here.) The Misiurewicz points along the axis cannot be seen, though, because they just form part of a straight line. Nevertheless, the structure of minibrots and synapses along the axis is the same as along all the other filaments.

In order to study this structure we need to look at a filament where the synapses can be identified. The illustrations opposite show the filament attached to lobe 3\4. This means that its synapses will have orders 3 and 4. Starting at the tip of the lobe we travel along a straight section, passing minibrots and (invisible) order 2 synapses along the way until we come to a prominent order 4 synapse. Taking the longest of the branches we trace a wiggly path through countless order 4 synapses interspersed with minibrots until we come to the first order 3 spiral synapse. After this, order 3 and order 4 synapses alternate – that is to say, between any two order 3 synapses you can find an infinite number of order 4 synapses and vice versa.

And, of course, between any pair of synapses you can find an infinite number of minibrots!





How Many Minibrots?

The question which has stimulated my exploration of the logistic equation is this – how many minibrots and synapses are there on the axis of the Mandelbrot Map (or what is the same thing, how many basins of attraction and Misiurewicz points are there in the logistic map)?

In my book on the Mandelbrot Map I argued that there was an uncountable infinity of minibrots along the axis. The argument goes something like this: If there was only one lobe (say of order 4), there would be an infinite number of order 4 synapses along the filament. With two lobes of order 4 and 3, there will be ∞^2 synapses (because there will be an infinite number of order 3 synapses between every order 4 synapse.) With *n* lobes stacked on each other the number of synapses will be ∞^n etc. Now we know that every filament is in fact attached to an infinite number of lobes (corresponding to the infinite series of bifurcations of the logistic map) so the total number of synapses along any filament will be ∞^∞ which is, of course, an uncountable number. This implies that there will be an uncountable number of minibrots too.

But on page 42 I have stated that, owing to the fact that the β point of every basin of attraction is an algebraic number, there will be a countable infinity of these and a countable infinity of Misiurewicz points.

As with many arguments, everything hinges on what you mean by the words you use. If you insist that a basin of attraction and a minibrot *must* have a finite extent – then there can only be a countable infinity of them. If, however, you let *n* tend to infinity, you admit the possibility that basins and minibrots can have zero width. And since the β point will be the solution to an equation of infinite order, β will no longer be algebraic but transcendental.

So all points can be put into one of four mutually exclusive categories

1. points which are inside a finite basin of attraction (or minibrot)
2. Feigenbaum points which are the limit of a period doubling sequence (or at the tip of a sequence of lobes)
3. Misiurewicz points with finite pre-period and finite period (synapses)
4. All the rest comprising all the zero width basins with infinite period, the Misiurewicz points with infinite pre-period and the truly aperiodic points (all of which are, of course, identical).

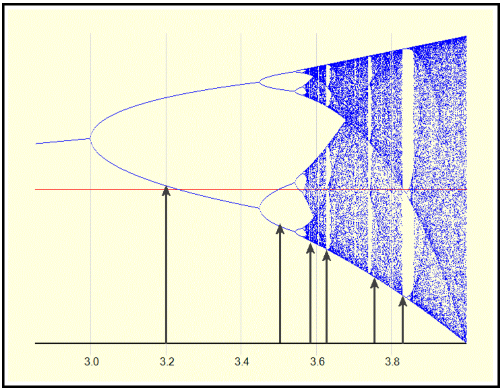
I have already hinted that I do not know whether any category 4 points exist but I strongly suspect that they do – in uncountable numbers. I also strongly suspect that they will remain forever hidden and that, while it may be easy to prove that a certain well defined (i.e. computable) transcendental number (like π) is *not* a category 4 number, it will be impossible to prove that it is.

May I be so bold as to call this the 'Linton Conjecture'?

Julia Graphs (i)

The pitchfork diagram shows us the behaviour of the critical orbit for each value of the parameter A. It is the Logistic Equations equivalent of the Mandelbrot Map which summarises the behaviour of the critical point for each value of $\boldsymbol{C}$ in the complex plane.

It is also of great interest to know what happens to the orbits of other starting points at different values of A. I call these graphs Julia graphs by analogy with the more familiar Julia maps in the complex plane. Six of these graphs are shown opposite and their positions in the chaos map are shown below:

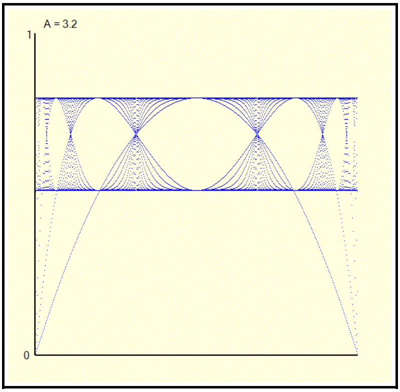


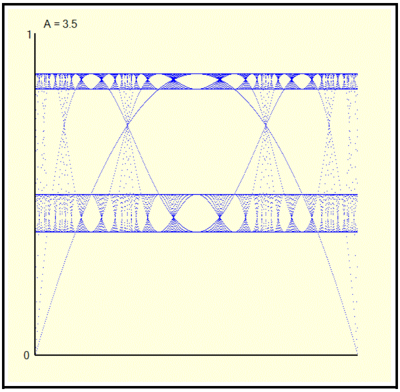
At $A = 2$ all points home in on a cycle of period 2 and at $A = 3.5$, the points home in on a cycle of period 4.

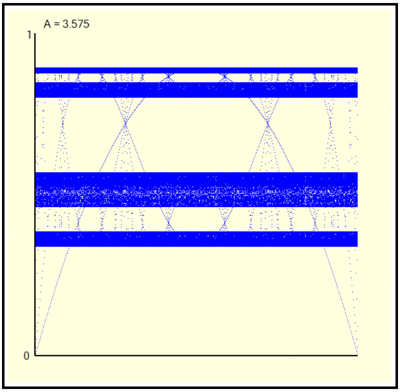
At $A = 3.575$ the first chaotic bands appear.

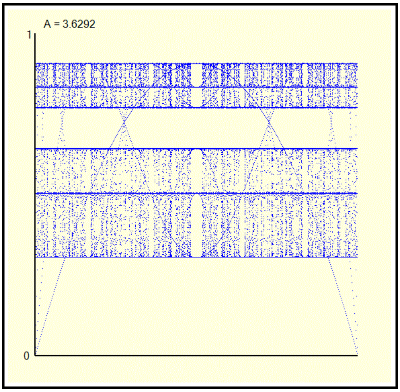
At $A = 3.6292$ there is a band of stability of period 6. Note that not all initial starting points find the stable cycle but the critical point $x = 0.5$ always does. The same is true of the last graph at $A = 3.834$ which is inside the period 3 band of stability.

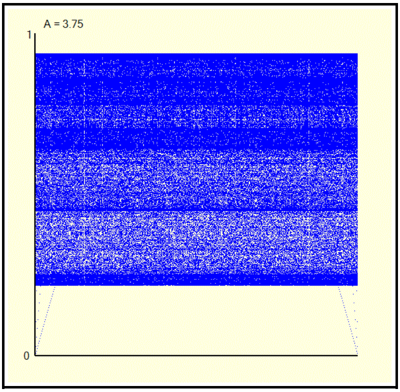
At $A = 3.75$ chaos appears to spread over the whole region but look very carefully and you may be able to see thin vertical white lines which represent individual values which remain stable for while.

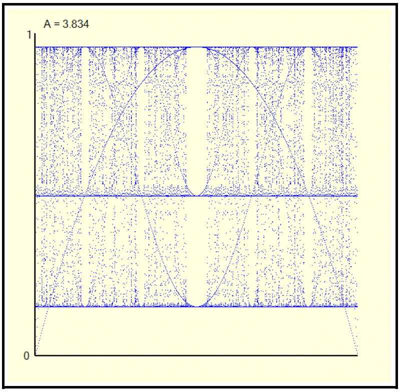

A = 3.2
1
0


A = 3.5
1
0


A = 3.575
1
0


A = 3.6292
1
0


A = 3.75
1
0


A = 3.834
1
0

JULIA GRAPHS (2)

In order to discover exactly what is happening at values like $A = 3.75$ where chaos appears to reign, it is instructive to pick out certain pairs of iterations. The first of the graphs on the opposite page shows the first and second order curves $f^1(x)$ and $f^2(x)$ which we met on page 10. It is easy to show that the two places where they cross (apart from 0 and 1, of course) are

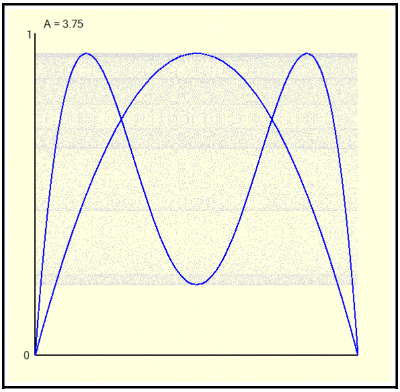
$$\frac{1}{A} \quad \text{and} \quad \frac{1-A}{A}$$

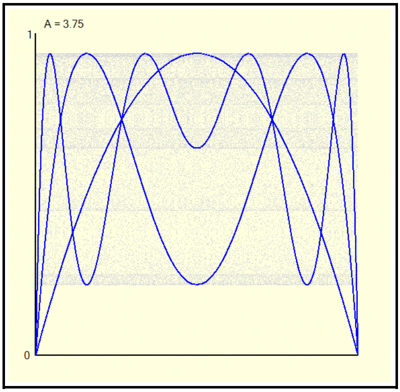
If we add the third order curve, we see that this curve also passes through these two points but adds another 4 crossing points with the first order curve. It also crosses the second order curve in two more different places.

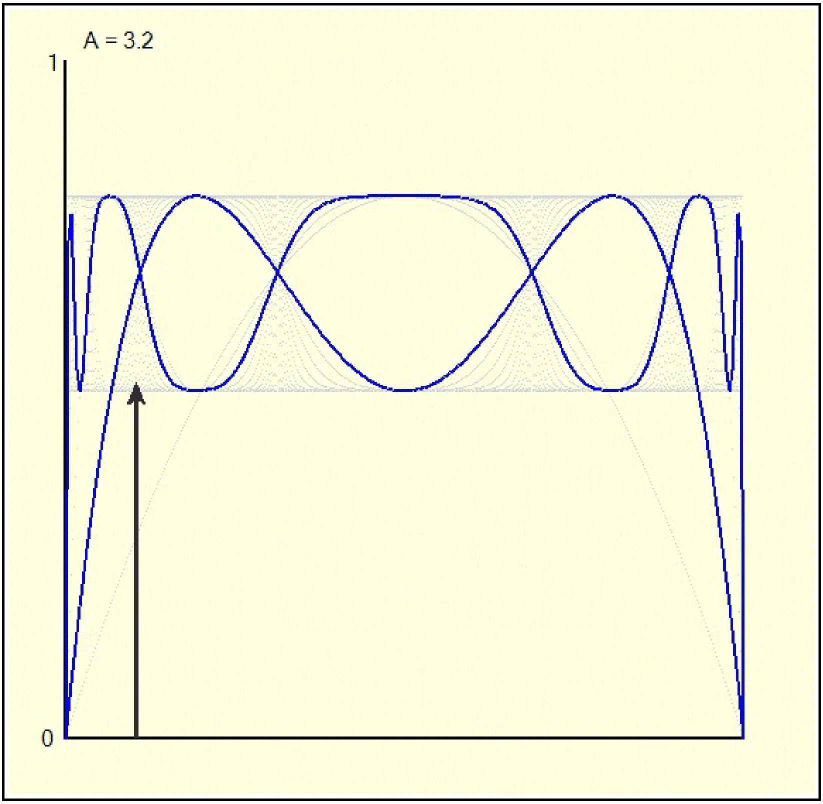
Now, whenever the n^{th} order curve crosses the m^{th} order curve $(n > m)$ then this orbit will enter a periodic cycle of order $p = n - m$. This is the exact analogy of the Misiurewicz points we met earlier. I do not know if they have been dignified with a special name so I will simply call them *periodic points*. We can immediately see that for any given value of A (in the chaotic region) there will a huge number of periodic points – but, of course, between then there will be a uncountable number of chaotic ones.

In fact these periodic points exist even outside the chaotic region. The third graph opposite shows the Julia graph of $A = 3.2$ with the second and fifth order iterations highlighted. At the point marked with the arrow, the fifth iteration is equal to the second and so this point will have a period of 3. But $A = 3.2$ is inside the period 2 band of stability so how can there be points with period 3?

The answer is that these periodic points are very rare and unstable; the slightest deviation will cause the orbit to home in on the stable cycle. This is why, if you want to find the stable cycles, it is important always to start from the critical point. If you start from a random point you may end up in a completely different cycle.


A = 3.75
1
0


A = 3.75
1
0


A = 3.2
1
0

THE ULTIMATE CHAOS

In order to find a period point with a given period p you must solve the equation $f^p(x) = x$. Alternatively you can simply see see where the p^{th} order iteration crosses the 0^{th} order line. (The 0^{th} order line is a 45° line from (0, 0) to (1, 1)) It follows immediately that for every value of A (>0) there will be at least 1 periodic point (not including $x = 0$) for every value of p. As A is increased, the number of crossing points (and therefore the number of periodic points) will increase dramatically and will reach a maximum at $A = 4$.

The Julia graph for $a = 4$ is shown on the opposite page and illustrates 800 iterations of each initial value. It can be seen to be crossed with several vertical white lines. To be honest, these are a bit of an artefact caused by the fact that the computers resolution is finite and some starting points will return to the exact same value after a while simply because the computer cannot calculate the difference. On the other hand, we know that there really are an infinite number of periodic points in the graph so the picture is not misleading – just wrong!

Now for virtually all values of A the only way to find out where a given starting point will end up after n iterations is to calculate each iteration; and since you can only do that to a finite resolution, it is rarely possible to prove by this means that a given starting point is periodic. But for the special case of $A = 4$, we actually have an analytical solution. The best way to describe this solution is to give an example. What is the value of $x = 0.1$ after 7 iterations?

Step 1: take the square root of x and calculate the angle θ_1 whose sin is root x. In this case $\theta_1 = 18.435°$.

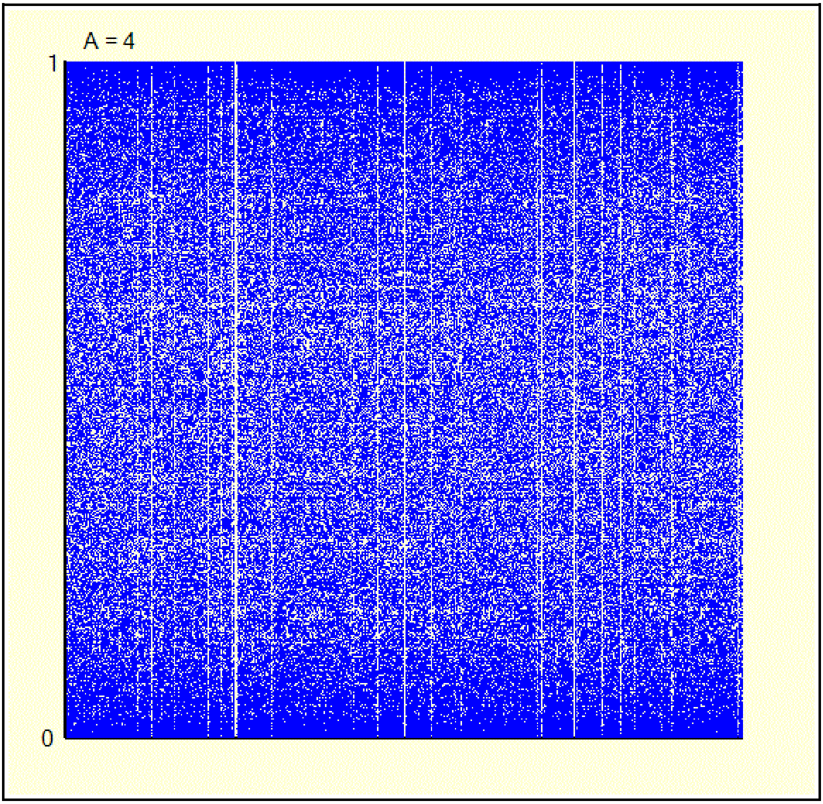
Step 2: multiply this angle by 2^n. $\theta_2 = 2359.7°$.

Step 3: take the sin of this angle and square it. $x_7 = 0.11334$

If you want a formula, this is it:

$$x_n = \sin^2\left(2^n \sin^{-1}\sqrt{(x_0)}\right)$$

To see why this formula works, see the panel opposite.



Consider the following facts:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\sin^2(2\theta) = 4\sin^2\theta\cos^2\theta$$
$$\sin^2(2\theta) = 4\sin^2\theta\,(1 - \sin^2\theta)$$
$$x_2 = 4x_1(1 - x_1)$$
$$\text{where } x_1 = \sin^2\theta$$
$$\text{and } x_2 = \sin^2 2\theta$$

Remarkably we see the logistic equation (with A = 4) emerging naturally from a simple substitution for x.

STRETCHING AND FOLDING

Notwithstanding the fact that we actually have a formula for the n^{th} iterate of any starting value, it is still correct to say that every value is chaotic – even those values which are periodic! 'Chaotic' does not mean 'unpredictable'. The defining characteristic of chaos is not unpredictability but extreme sensitivity to initial conditions. Take for example, the starting value $x = 0.25$. The orbit of this value is:

$$0.25 \ \rightarrow \ 0.75 \ \rightarrow \ 0.75 \ \rightarrow \ \ldots$$

and every term thereafter is 0.75.

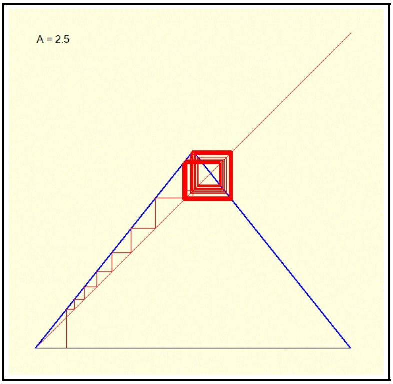
Now if we calculate the orbit of 0.250001, we find that after only 20 iterations, we are all over the place. The reason is simple. If $x = 0.25$, the angle $\theta_1 = 30°$ whereas if $x = -.250001$, $\theta_1 = 30.000066°$. The difference is a miniscule 0.000066°. But when we multiply this angle by 2^{20} this difference becomes big enough to throw the calculations out completely.

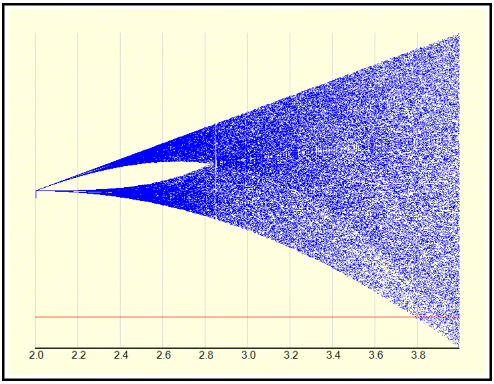
The clue is in the fact that the number of iterations n appears as an exponent in the formula and is why the discrepancy increases exponentially. It is easy to show that in this case the Lyapunov exponent is equal to $\log(2) = 0.693$.

It is also clear that if θ_1 is any rational fraction of a circle, there will always (?) be a value of n which makes $\theta_2 = \theta_1$. For example, if $\theta_1 = 1/9$ then after 6 iterations $\theta_2 = 2^6 \times 1/9 = 64/9 = 7 + 1/9 = 1/9$.

Another way of looking at the process is to imagine that the numbers from 0 – 1 are stretched to double their length and then folded back on themselves. When this process is done repeatedly, the result is a structure like flaky pastry with numbers originally close together now far apart and vice versa.

Now the logistic equation uses a smooth quadratic function to do the stretching and folding but we can investigate an even simpler system in which the stretching in linear. The function we need is called the tent function shown opposite with its chaos graph..





There are no 'islands of stability' because nowhere does the function have zero gradient so chaos reigns supreme. At the limit ($A = 4$) we have the simple formula[6]

$$x_n = residue(2^n x_0)$$

whose behaviour is most easily seen if we write x_0 in binary.

Multiplying a binary number by 2^n is equivalent to shifting the binary number n places to the left so if $x_0 = 0.011010100111$, say, than after 5 iterations we will reach 1101.0100111 whose residue is 0.0100111. Now it is immediately obvious that any rational value of x_0 will eventually either reach 0 (when the binary number has run out of digits) or enter a periodic cycle (when the binary representation repeats). Also it is clear that if x_0 is irrational, then the orbit will never ever repeat.

This example should finally dispel the idea that there is any 'magic' in chaos. Whatever function you choose to iterate, the orbit of a particular staring value is completely encoded in the value itself. If you start with a simple number, the orbit will be simple; but if you start with a complicated number, the orbit will be complicated too.

Carr Bank, May 2018

6 Actually this formula is appropriate to a slightly different function, the 'sawtooth' function but the principle remains the same

Most of the illustrations in this book were generated using a program called 'Chaos Explorer' written by the author and available on his website: www.jolinton.co.uk.