The Garbage Problem

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Abstract: A simple analysis of elliptical orbits leads to an explanation of the reasons why you cannot simply throw away garbage in space and for the stability or otherwise of the Lagrange points.

The International Space Station orbits the Earth at a height of 400 km above the Earth's surface at a speed of 7.67 km s^{-1} taking 92.65 minutes to complete a whole orbit. It has been almost continuously occupied for 18 years and inevitably this has resulted in a large quantity of waste both organic and mineral which has to be disposed of. All of this waste has been returned to Earth in the supply vessels used to maintain the station but you might well ask the question – why not just throw it away into space? And if you didn't want to litter the Solar System with garbage, why not project it downwards (i.e. towards Earth) where it will burn up harmlessly in the Earth's atmosphere?

The reason is pretty obvious. If we assume that the orbit of the ISS is circular, then if we eject the can of garbage in the same direction as the velocity of the station, then all we have done is to put the garbage can into a slightly elliptical orbit with a slightly longer period than the station. Looking out of the observation windows, the crew will see the garbage can slow down as it rises; 46 minutes after launch it will be 'above' them travelling backwards and another 46 minutes later it will be catching up with them from behind! You can't get rid of garbage that easily.

It is instructive to work out what happens if the garbage can is ejected backwards, 'upwards', 'downwards' or sideways. Although the latter case is a bit different, the net result is the same – the garbage enters a very similar orbit to that of the space station and periodically it will return to haunt them.

Lets work out the details of the problem assuming that the garbage is ejected 'forwards' with a speed of U relative to the station which is travelling at a speed V with an orbital radius of R. It follows that the period of the circular orbit is given by:

$$T_{ISS} = 2\pi R/V \tag{1}$$

Working out the period of an elliptic orbit whose speed at perigee (closest to Earth) is $V + U$ is a bit more tricky and we need to use some of the mathematics relating to ellipses (see *fig. 1*).

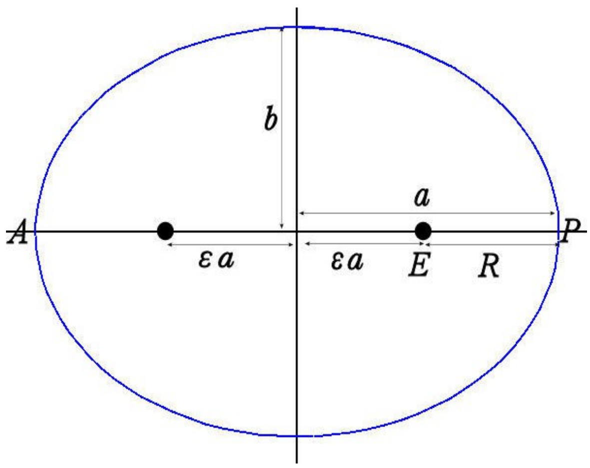

fig. 1: The dimensions of an ellipse

An ellipse has two semi-axes of length a and b and two focal points. The distance from each focal point to the centre is εa where ε is the eccentricity of the ellipse. The relation between a, b and ε is:

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}} \tag{2}$$

It is easy to see that the eccentricity of a circle (when $a = b$) is zero and that since a is (by definition) always greater than b, ε is always within the range 0 - 1. If the Earth is at the focus E and a satellite is launched at perigee P then it starts at a distance $a - \varepsilon a$ above the Earth and half an orbit later it rises to $a + \varepsilon a$. By the law of conservation of angular momentum, the speed at apogee (v') is related to the speed at perigee v by the equation:

$$v'(a + \epsilon a) = v(a - \epsilon a) \tag{3}$$

Next we equate the total energy of the satellite at the two extremes of its orbit:

$$\frac{1}{2} m v'^2 - \frac{GMm}{a(1 + \epsilon)} = \frac{1}{2} m v^2 - \frac{GMm}{a(1 - \epsilon)} \tag{4}$$

$$\frac{GM}{a}\left(\frac{1}{1 - \epsilon} - \frac{1}{1 + \epsilon}\right) = \frac{1}{2} v^2 \left(1 - \left(\frac{1 - \epsilon}{1 + \epsilon}\right)^2\right) \tag{5}$$

from which we deduce that:

$$v^2 = \frac{GM}{a}\left(\frac{1 + \epsilon}{1 - \epsilon}\right) \tag{6}$$

Now, since

$$a(1 - \epsilon) = R \tag{7}$$

we find that

$$\begin{aligned} \epsilon &= \frac{Rv^2}{GM} - 1 \\ a &= R/(1 - \epsilon) \end{aligned} \tag{8}$$

Together, equations (8) enable us to calculate the dimensions of the ellipse when a satellite is launched with any speed v parallel to the Earth's surface at a radial distance R.

What we need now is an expression for the period of its orbit. This is easily calculated using Kepler's second law – that any satellite sweeps out equal areas in equal times.

We know that at perigee, the satellite sweeps out an area equal to $\frac{1}{2}Rv$ every second so the period of the orbit T_{GC} is simply the area of the ellipse (πab) divided by this hence:

$$T_{GC} = \frac{2\pi ab}{Rv} \tag{9}$$

Combining equation (2), (6), (8) and (9) leads us to the following elegant and remarkable result:

$$T_{GC} = 2\pi\sqrt{\frac{a^3}{GM}} \tag{10}$$

(Essentially this is what Kepler discovered and encapsulated in his third law. The period of a planet round a sun depends only on the *diameter* of its orbit ($2a$), not its eccentricity.)

Now it would be nice if we could find out exactly where the garbage can ends up after one orbit of the space station. What we need is $\Delta T = T_{GC} - T_{ISS}$

$$\Delta T = 2\pi\left(\sqrt{\frac{a^3}{GM}} - \sqrt{\frac{R^3}{GM}}\right) \tag{11}$$

which we can simplify using equation (8b) and the fact that ε is very small to:

$$\Delta T = 2\pi\sqrt{\frac{R^3}{GM}}.\frac{3}{2}\epsilon \tag{12}$$

We can use some similar approximations (with $v = V + U$) to reduce equation (8a) to:

$$\epsilon = \frac{2\mathrm{RUV}}{GM} \tag{13}$$

so, combining these together we get:

$$\Delta T = 2\pi\sqrt{\frac{R^3}{GM^3}}.3\mathrm{RUV} \tag{14}$$

We can get rid of the *G*'s and *M*'s by remembering that

$$\frac{V^2}{R} = \frac{GM}{R^2} \tag{15}$$

which gives us

$$\Delta T = 2\pi.\frac{3\mathrm{RU}}{V^2} \tag{16}$$

hence using equation (1)

$$\Delta T = \frac{3\mathrm{T}_{ISS}U}{V} \tag{17}$$

It is time we put in some figures. With T_{ISS} =5600 s, U = 1 ms^{-1} and V = 7670 ms^{-1}, ΔT works out to be 2.2 s. i.e. the garbage can goes into an orbit whose period is 2.2s longer than that of the space station. What this means is that when the garbage can returns exactly to the place from which it was launched, the space station will have already passed this point and be 7670 × 2.2 = 15340 m ahead. You might conclude that this was a safe margin and that the space station is not in any danger of being hit by its own garbage. Not so. While it is true that every orbit, the garbage can lags behind by 15 km, it is equally obvious that in 5600/2.2 = 2545 orbits the space station will be catching it up from the other side. And the ISS does 2500 orbits in only 164 days!

In point of fact, there is already so much space junk up there already that the ISS has had to take evasive action to avoid it on many occasions.

Docking

A similar situation arises when you are attempting to dock with the space station. Suppose that the launch vehicle places you in exactly the same orbit as the space station but you find that you are 15 km behind. What do you do?

If you fire your rockets to give you an extra speed of 1 ms^{-1} towards your target, you will find yourself rising into a higher, slower, orbit and 93 minutes later you will find yourself 30 km *behind* the station. What you must do, of course, is fire your thrusters *backwards* to slow you down by 1 ms$^{-1.}$ Your craft will fall, speed up, and – hey presto – 93 minutes later you will find yourself next to your object. (Don't forget to fire your thrusters forwards again at this point to match the speed of the station!)

The Coriolis force

Another way of looking at the problem is to consider the whole situation from a rotating frame of reference in which the space station is stationary. From this point of view the reason why the garbage can rises and circles over the space station is because of a fictitious force called the Coriolis force. For an object moving at right angles to the axis of the rotating frame this force is proportional to the velocity of the object (relative to the rotating frame) and acts at right angles to both the velocity and the axis of the frame of reference. This is not the whole story though because if the Coriolis force was the only one acting on the garbage, it would move in a circle round the space station and collide with the station on the first orbit. We must not forget the other forces acting in a rotating frame – first, obviously, the force of gravity and second the *centrifugal* force. The latter is an outward force which is proportional to R the distance from the centre of rotation while gravity is, of course, inward and inversely proportional to R^2. At the radius of the orbit of the space station, these two forces exactly cancel each other out but on either side, the forces do not balance. If you stray outside the correct orbit, the combination of gravity and centrifugal force will push you further away and if you stray inside it you will be pushed further in. This simple analysis would seem to suggest that orbits are unstable and that a slight perturbation from the exact circular orbit will result in object either being flung into space or spiralling down to Earth. But we know that this is not the case.

In a previous article [reference here] I introduced the idea of centrifugal potential. From the point of view of a rotating frame of reference, we may map a combination of the gravitational potential and the centrifugal potential at any point using the following equation:

$$Potential = -\frac{1}{2}R^2\omega^2 - \frac{GM}{R} \tag{18}$$

and you can visualise the result as if it was a deep funnel-shaped crater in a dome-shaped mountain (see *fig 2*).

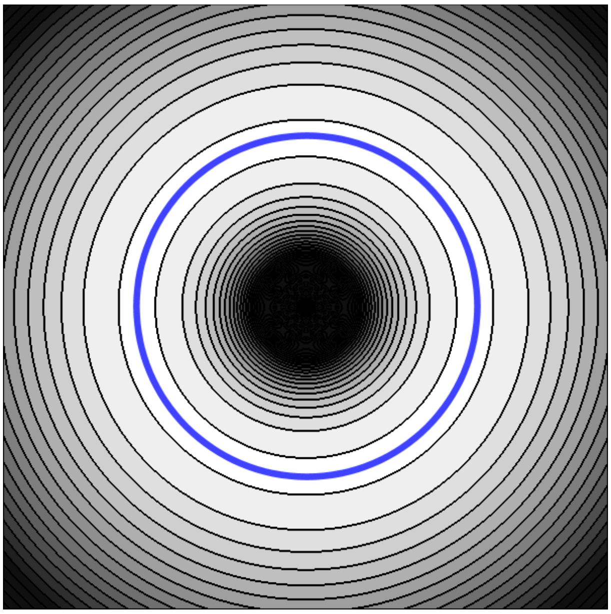

fig. 2: Potential map round a planet

The blue line is the locus of all the equilibrium points where gravity and centrifugal force balance but if you tried to place a ball bearing on the lip of such a potential field, it would soon fall off, either down the mountainside or into the crater. (N.B. The blue line is not the *orbit* of a satellite. In the rotating frame of reference a satellite in a circular orbit is *stationary* at a single point on the line.)

So how can elliptical orbits exist and why are they stable?

The reason is the Coriolis force. As soon as you ball bearing (or satellite) falls off to one side it begins to acquire speed. The faster it goes, though, the greater the Coriolis force deflecting it sideways. Eventually it curves so much that it climbs back up the hill, reaching the top somewhere near where it started. The result is a stable orbit that loops round and round as shown in *fig. 3* which shows the orbit of a garbage can projected forwards (i.e. upwards on the map), each loop carrying it further and further behind the space station (the blue dot).

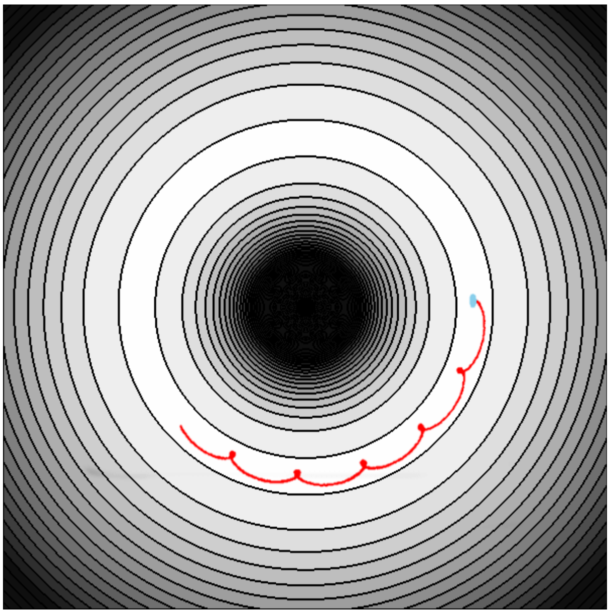

fig. 3: Potential map round a planet with a garbage can orbit

The stability of the Lagrange points

We are now in a position to understand the stability, or otherwise, of the Lagrange points of a binary system consisting of a star and a smaller planet as discussed in the previously mentioned article. The potential map of such a two body system is a bit more complicated and it looks like this:

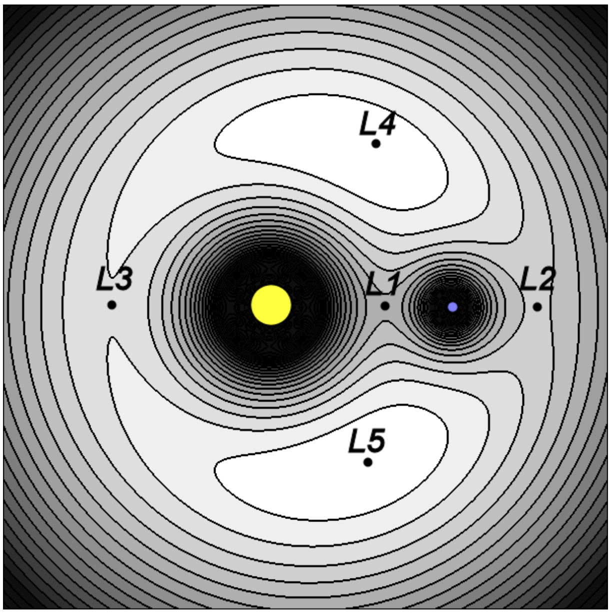

fig. 4: Potential map around a binary system

There are five Lagrange points and at each of these points the potential surface is horizontal. It follows that a satellite placed at any of them will be in equilibrium but only the orbits at L4 and L5 will be stable against small perturbations. The argument is the same as the one developed above. If the satellite moves off the local summit in any direction, the Coriolis force will steer it back up again. What is more, it is easy to see that, launched at the right speed, a satellite can 'orbit around' either L4 and L5 in an approximately circular stable orbit in much the same way that it can orbit a planet.

This is not the case at a saddle point. A satellite given a small nudge downhill towards either the star or the planet will gather speed and attempt to climb back up again but it will not return exactly where it started from and because of the shape of the slopes it will soon be flung out of orbit. Nevertheless, there are many advantages in placing satellites at or near L1 and L2 in particular and with sufficient fuel on board it is possible to maintain a satellite there for its useful life.