Cosmological expansion

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Abstract. As teachers, we want to encourage our students to ask searching questions like 'How old is the universe?', 'How much of the universe can we actually see?' and 'How far away is the Cosmic Microwave Background radiation?' But how many of us can honestly say we know the answers? And even if we know the answers – how are we going to respond to the student's next obvious question – 'But Sir; how can the Cosmic Microwave Background radiation be 90 billion light years away when the universe is only 13.6 billion years old?'

In this article, using some straightforward mathematics I derive some interesting formulae which will not only provide reasonably accurate numerical answers, but, more importantly, give much needed insight into the paradoxes which abound when applying General Relativity to the universe as a whole.

Our universe

Of all the remarkable things about the universe we live in, perhaps the most remarkable is that we can see it at all. Just suppose our planet was shrouded, like Venus, in thick cloud. night and day, the seasons and the tides, all would be a complete mystery. It is debatable whether physics would ever have got off the ground. It is probable that our understanding of the nature of gravity would have been limited to an Aristotelian tendency for things to fall into their natural place.

Suppose that our planet was cloud free but space was filled with interstellar dust so that we could only see the solar system and our neighbouring stars. Isaac Newton would have had sufficient information to work out his famous laws but would we ever have deduced that the stars were part of a vast whirlpool we call our galaxy? Einstein might have been able to formulate his theories of Special and General Relativity but Hubble would never have discovered that the universe was expanding.

So when next you gaze on the wonder that is the night sky or when you open a book of photos taken from the HST, you are bound to feel a sense of profound gratitude as well as awe; for if the universe had been ever so slightly different, we might have had no knowledge of it at all. As it is, advances in observational technologies and theoretical astronomy over the last three decades have given us an unprecedented amount of information about our universe but, as always, this information has often served to raise more questions than provide answers. As far as we can tell [1], on a sufficiently large scale the universe we live in is isotropic (i.e. it looks the same in all directions) and is homogeneous (i.e. is basically the same everywhere). But is it infinite or finite? Is its expansion accelerating or decelerating? Is it dominated by matter or dark energy? Is there enough matter in it to stop the expansion or will it expand for ever? How much of it can we see? No one knows for sure.

But we are getting close.

Einstein's theory of General Relativity has withstood every test that has been thrown at it for nearly a century and although there have been many attempts to modify it, it is still the best we have got at the moment. Unfortunately, though, for mere mortals like me, it has to be admitted that the full theory is too difficult to understand, and this is not helped by the fact that certain consequences of the theory seem too bizarre to accept.

All is not lost, however, because we do not need the full theory to understand how our universe is likely to behave for two reasons: a) it is a fairly simple universe being isotropic and homogeneous and b) some clever mathematicians called Friedmann, Lemaître, Robertson and Walker worked out the solution to Einstein's field equations for an isotropic, homogeneous universe in the 1920's and 30's and presented them in a form which I can understand. Here they are.

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{1}{3}\Lambda + \frac{8\pi G}{3}\rho - k\frac{c^2}{a^2}$$
$$\frac{\ddot{a}}{a} = \frac{1}{3}\Lambda - \frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) \tag{1}$$

Essentially, these two differential equations define an arbitrary time dependent term $a(t)$ which determines the 'size' or scale of the universe. The other constants in the equations are as follows:

- H is the Hubble constant
- Λ is called the cosmological constant. Einstein introduced it originally in order to make his universe static but then, when it was discovered that the universe was expanding, he withdrew it calling it 'the greatest blunder in my life' (But see [2] for a fuller account). Recently though it has become popular again in the guise of 'dark energy'.
- G is the Newtonian Gravitation constant – 1.67 x 10^{-11} kg^{-1} m^{-3} s^{-2}
- ρ is the mean density of matter in the universe (NB this term is also time dependent and therefore also depends on a)
- P is the pressure due to radiation in the universe (also dependent on a)
- c is the velocity of light
- k is a constant which can take the values -1, 0 or 1. If k = -1, the universe has an overall negative curvature. If k = 1, the universe is positively curved while k = 0 describes a 'flat' i.e. Euclidean universe.

The first equation tells us how to calculate the rate of expansion of the universe (ie the Hubble constant). It contains a very surprising consequence. Matter causes the universe to expand! (Positive values of ρ increase H). This is important because it helps us to answer the question 'what causes the universe to expand?' According to General Relativity, the expansion of the universe is simply a natural result of the fact that it contains matter.

The second equation is the equivalent of Newton's second law as applied to the universe as a whole. $\frac{\ddot{a}}{a}$ represents the rate of change of the rate of expansion of the universe (ie its acceleration). It should be noted that the contribution of matter and radiation $\left(\rho+3\frac{P}{c^2}\right)$ is negative (ie it causes a deceleration). This can be interpreted classically as the attractive gravitational force causing the expansion of the universe to slow down. The effect of Λ (the cosmological constant) however, results in an acceleration.

Now as far as we can tell, the radiation pressure in the universe at the moment is much less than the density of matter in it [1] (though the situation was the other way round in the first few seconds of the universe). This means that we can put $P = 0$.

Also, as far as we can tell, the universe appears to be almost exactly flat [1]. This means that we can ignore the kc^2/a^2 term as well.

If we assume that the cosmological constant Λ = 0, the equations simplify even further and admit of a straightforward solution which is

$$a(t) = a_0 t^{2/3} \tag{2}$$

(This universe starts with a 'Big Bang' at $t = 0$ and the constant of proportionality a_0 is arbitrary and can, if desired, be chosen to make a equal to unity at the present time.)

The Greek for two-thirds is δυο τριτα so I shall call this universe the **duotritaic universe**.

On the other hand, if we live in a universe which is dominated by the cosmological constant Λ, we can put $\rho = 0$ and the solution to the equations has the form

$$a(t) = a_0 e^{t/\tau} \tag{3}$$

which I shall call the **exponential universe**.

(This universe in infinitely old and has no 'Big Bang'. As before, a_0 and τ are arbitrary constants. τ is the time taken for the universe to expand by a factor of e and a_0 can be chosen to make a equal to unity at the present time.)

I shall also consider an intermediate universe in which

$$a(t) = a_0 t \tag{4}$$

which I shall call the **linear universe**. (This universe is basically empty!)

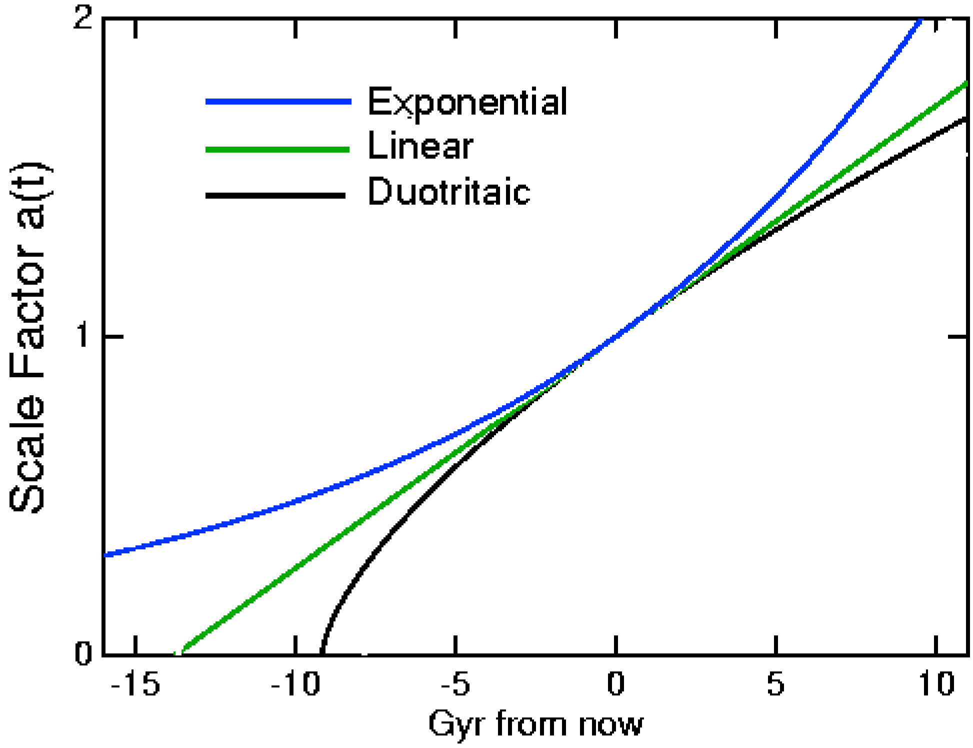

Figure 1: The expansion of the universe according to three different models. All three models are 'flat' (ie $k = 0$). The exponential universe is dominated by the cosmological constant; the duotritaic universe is dominated by matter.

The graphs shown in figure 1 (which is adapted from 'Ned Wright's Cosmology Tutorial' [3]) shows the behaviour of the three universes. (Please do not confuse this with a similar diagram which describes the differing behaviour of 'open' and 'closed' matter-dominated (ie duotritaic) universes. **All these universes are 'flat'**) The exponential universe is, of course, infinitely old. The currently accepted value of the Hubble constant H = 71 (km/s) / Mpc. If the universe is expanding linearly, its age is simply the reciprocal of this figure – i.e. 13.6 billion years. As we shall shortly prove, the age of a duotritaic universe is 2/3[rds] of this i.e. 9.2 billion years.

But what actually **is** $a(t)$? This the issue which causes most of the confusion.

One analogy which is often used is that of a balloon being blown up and that $a(t)$ is implicitly identified with the radius of the balloon. This gives the impression that if $a(t)$ is positive and finite, the universe must also be finite and have positive 'curvature'. This is not the case. It is perfectly possible for our universe to be infinite and flat and still have a finite a which may or may not be changing with time. a is in fact a 'scale factor' not a radius. But for many purposes it can be useful to think of it as a 'radius' in some other dimension outside our universe. In particular, it is a useful fiction when trying to work out what happens to a photon as it crosses the vast distances of inter-galactic space. It is worth emphasising again that the actual value of a is unimportant. It is only the way that a changes with time which has any relevance.

Comoving observers

There is another serious source of confusion which makes people frightened of applying General Relativity to the whole universe.

Ever since Copernicus demoted the Earth from its central position in the cosmos we have, bit by bit, thrown away every bit of anthropocentrism. The Special theory of Relativity even discarded the notion that there was a unique coordinate system of space and time on which we could all agree. So what hope is there in General Relativity of talking about, for example, the distance between us and a distant galaxy *at this instant* if nobody agrees on a measure of distance or even a point in time called *now*?

We need not despair. There *is* a coordinate system on which all (so-called comoving) observers in the universe can agree but I must first define what is meant by a comoving observer.

When we look out into space we see galaxies receding in all directions. If our universe is isotropic (as assumed) then the red shift of galaxies at a certain distance should be the same in all directions. If, however, we were to observe a consistently greater red shift in one particular direction and a corresponding blue shift in the opposite direction, we would conclude that we were in fact moving through the universe towards the latter point. If we start up our rocket motors and point our rocket in the direction of the red-shifted galaxies, we can adjust our speed so that the bias completely disappears. We are now actually stationary – or rather comoving with – the universe.

(Our sun does in fact have a proper motion which is currently about 370 km/s towards the constellation of Virgo but for this purposes of this paper we shall assume that all the galaxies in the universe are comoving and, like the spots on the expanding balloon, are, in an important sense, stationary with respect to the universe.)

Now here is the important point. Since all comoving observers are stationary with respect to the universe (notwithstanding the fact that the distances between them are changing and they look as if they are moving with respect to each other) all such observers share the same *now* and possess metre rulers which are the same *length*. To put it another way, if I measure the distance to a certain galaxy as S' at a time T' after the Big Bang, then if there is an alien observer on that galaxy doing the same measurements as me *at this instant*, it will get exactly the same results as me.

Universal expansion

So, following Sir Arthur Eddington, let us liken the expansion of our universe to the expansion of a balloon in order to explain how all the galaxies can move apart from each other without their being any unique point on the surface of the balloon where the expansion started. The balloon is a 2-dimensional surface embedded in a 3-dimensional medium. We are asked to imagine our universe as a 3-D space embedded in a 4-D medium.

So far, so good.

Now suppose that the balloon expands so that its radius R changes with time according to some function $R = a(t)$, and consider an ant crawling round the balloon at a speed v. (See figure 2.) How far does he get in a given time? Under what conditions can he walk right round the balloon and return to his starting point? What happens if, for example, the ant's speed v is less than the rate at which R increases?

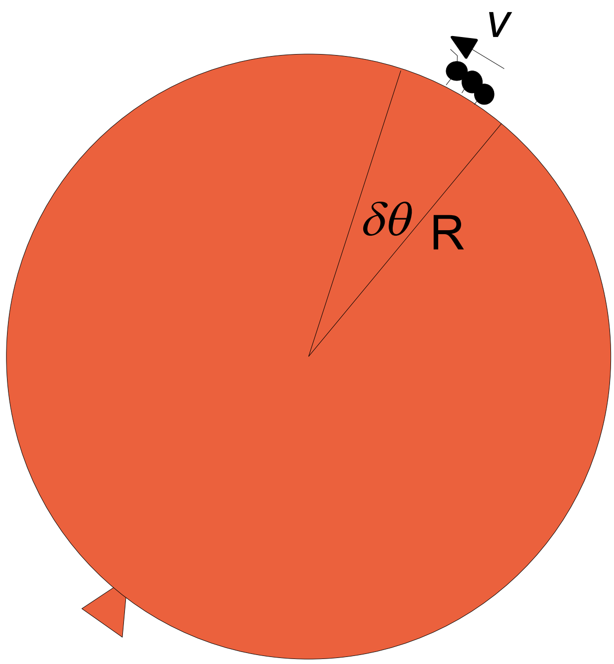

Figure 2: An ant walking round an expanding balloon.

Instead of measuring his progress in terms of distance we must consider his angular progress.

In a short tine δt he moves a distance along the circle $v\delta t = R\delta\theta$. But since $R = a(t)$,

$$v\,\delta t \;=\; a(t)\,\delta\theta \tag{5}$$

$$\theta \;=\; v\int_{T_S}^{T_0} \frac{1}{a(t)}\,dt \tag{6}$$

where T_S is the time when the ant *started* walking and T_0 is the time when he *stops* walking. (It is a little confusing to use the subscripts this way round but the reason is that T_0 is often interpreted as the time *now*. Just bear in mind that T_S is *less* than T_0)

So the total distance the ant appears to travel when he gets to his destination S is

$$S \;=\; a(T_0)\,\theta \;=\; v\,a(T_0)\int_{T_S}^{T_0} \frac{1}{a(t)}\,dt \tag{7}$$

This is an important result which can be applied even when the 'radius' of the balloon is infinite. Just imagine a flat infinite sheet of rubber which is being stretched according to some scale factor a. The ant still has a velocity v with respect to the sheet at all times, but he is also being carried along by the expansion at the same time.

Now let's calculate how far the ant gets in each of our three universes.

In the **exponential universe** where $a(t) \;=\; a_0 e^{t/\tau}$ the integral looks like this

$$S \;=\; v e^{t/\tau} T_0 \int_{T_S}^{T_0} e^{-t/\tau}\,dt \tag{8}$$

and has the following solution: $$S \;=\; v\tau\,(e^{(T_0 - T_S)/\tau} \;-\; 1) \tag{9}$$

(It is clear that, while the distance the ant travels from his starting point increases exponentially, he can only return to his starting point if $v >= 2\pi\, a_0 / \tau$.)

In the context of a three-dimensional expanding universe – even if it is infinite in extent – the same formula applies because all the references to the extrinsic properties of the space (θ and a_0) have disappeared. It therefore follows that in an exponentially expanding universe - i.e. one dominated by a large cosmological constant, the effective distance travelled by a photon in a time ΔT, starting out at a time t_0 is:

$$S = c\tau(e^{\Delta T/\tau} - 1) \quad (10)$$

When ΔT is much smaller than t_0, the formula reduces – as, of course, it must – to $S = c\,\Delta T$. On the other hand, if ΔT is large, the distance travelled by the photon can be much greater than this as it is 'carried along' by the expansion of the universe itself.

In the **linearly expanding universe**, where $a(t) = a_0 t$ the integral is

$$S = vT_0 \int_{T_S}^{T_0} \frac{1}{t} dt \quad (11)$$

whose solution is:

$$S = vT_0 \ln \frac{T_0}{T_S} \quad (12)$$

Lets look at this formula carefully.

In the first place it is immediately obvious that the equation blows up to infinity if $T_S = 0$ so the formula only gives sensible results if the ant starts his walk a short time after the time that the expansion starts.

But what is really surprising is the angle through which the ant travels as T_0 increases. Since the logarithm function has no upper limit, the ant can make as many revolutions of the balloon as he likes, regardless of the slowness of his walk or the speed of expansion of the balloon! For example, if $v = a_0$ then we can rearrange the equation to get

$$T_0 = T_S e^{\theta} \quad (13)$$

so to walk round the whole balloon (so $\theta = 2\pi$) starting 1 minute after the start would take just 535 minutes (9 hours).

Relating this formula to a photon in a real universe we have:

$$S = cT_0 \ln \frac{T_0}{T_S} \quad (14)$$

We are not allowed to put $T_S = 0$ in this equation but if we set off a flash of light at $T_S = 1$, the photons will eventually reach every part of the universe.

In the **duotritaic universe** where $a(t) = a_0 t^{2/3}$ the integral is

$$S = vT_0^{2/3} \int_{T_S}^{T_0} t^{-2/3} dt \quad (15)$$

$$S = 3vT_0^{2/3}(T_0^{1/3} - T_S^{1/3}) \quad (16)$$

$$S = 3vT_0\left(1 - \sqrt[3]{\frac{T_S}{T_0}}\right) \quad (17)$$

Or, in the case of a photon:

$$S = 3cT_0\left(1 - \sqrt[3]{\frac{T_S}{T_0}}\right) \quad (18)$$

What this means is that the most distant photon that we can see (i.e. one which started out at $T_S \approx 0$) started out from a point which is now at a distance of $3cT_0$. If we live in a universe which is matter dominated and $T_0 = 9.2$ Gyr (the reason for this choice will become clear later) the cosmic background radiation or CMB is therefore 3 x 9.2 = 27.6 Glyr away from us at this moment – though it has to be admitted that this statement is very misleading. It does not mean, for example, that the photon has travelled faster than light. Nor does it mean that the universe has to be at least 27.6 Gyr old. Nor does it mean that the photon has travelled 27.6 Glyr. All very confusing! It simply means exactly what it says. The CMB is 27.6 Glyr away from us *at this moment*. Period.

Red shift

It is common knowledge now that the more distant a galaxy is, the more its light is red-shifted. It will often be stated that this is due to the Doppler shift and is evidence that the galaxies are moving away from us at high speed. I believe this picture is seriously misleading to a novice and that it is much better to regard the distant galaxies as being stationary (i.e. comoving) and that the red shift is caused by the photons wavelength being stretched by the universal expansion which takes place while the photon is in transit.

Astronomers define red shift z as

$$z = \frac{\lambda'}{\lambda} - 1 \tag{19}$$

In what follows, we shall find the following physical definition more useful

$$Z = \frac{\lambda'}{\lambda} = z + 1 \tag{20}$$

Since, as we have noted, photons partake in the cosmological expansion of the universe

$$\boxed{Z = \frac{\lambda'}{\lambda} = \frac{a(T_0)}{a(T_S)}} \tag{21}$$

This is an extremely important idea. Z is simply a measure of how much the universe has expanded in total between the time the photon was emitted and now. e.g. for a galaxy whose red shift z is 2, the expansion factor Z is equal to 3 and the universe is now three times larger than it was when the photon started out. The red shift of the CMB is over 1000. This means that the universe is over 1000 times bigger now than it was at the time of the recombination event. Incidentally, it also means that the universe was a billion times more dense than it is now (which is still pretty sparse!)

Lets see how S (ie the distance between us and the object which emitted the photon at this exact moment) depends on Z in our three universes.

In the **exponential universe**:

$$Z = \frac{a_0 e^{T_0/\tau}}{a_0 e^{T/\tau}} = e^{(T_0 - T)/\tau} = e^{\Delta T/\tau} \tag{22}$$

From equation (9) we have

$$S = c\tau(e^{\Delta T/\tau} - 1) \tag{23}$$

so eliminating ΔT we get:

$$\boxed{S = c\tau(Z-1) = c\tau z} \tag{24}$$

(Here the astronomer triumphs over the physicist in that the distance to the distant galaxy is directly proportional to the astronomical red shift, not the expansion factor.)

We can also show that the time taken for the photon to reach us ΔT ($= T_0 - T_S$) is given by

$$\Delta T = \tau \ln Z \tag{25}$$

Now in any universe which starts with a Big Bang the expansion factor Z (and therefore the red shift z) can be arbitrarily large so when Z is infinite, both S and ΔT are infinite too. This means that in an exponential universe, the Big Bang itself is infinitely far away and happened infinitely long ago. ie the 'observable universe' is infinitely large and the universe is infinitely old.

In a universe which is expanding at a constant rate – the **linear universe:**

$$Z = \frac{T_0}{T_S} \tag{26}$$

so from equation (14)

$$S = c T_0 \ln Z \tag{27}$$

and

$$\Delta T = T_0 (1 - 1/Z) \tag{28}$$

As we have seen earlier, *Z* can be arbitrarily large so whereas the size of the whole universe (both observable and unobservable) is therefore infinite, its age is finite.

In the **duotritaic universe**

$$Z = \frac{a_0 T_0^{2/3}}{a_0 T^{2/3}} = \left(\frac{T_0}{T}\right)^{2/3} \tag{29}$$

so from equation (18)

$$S = 3 c T_0 (1 - \sqrt{1/Z}) \tag{30}$$

and

$$\Delta T = T_0 (1 - 1/Z^{3/2}) \tag{31}$$

Here we see even more clearly that in this universe, not only is the universe finite in age, it is also finite in extent.

How far away is that galaxy?

When we see a distant galaxy whose red shift Z = 4, for example, it is natural to ask the question how far away is it?

There are three different answers to this question – or rather, there are three quite different questions which we may ask.

1. How far away is that galaxy *now*? - the answer is *S*
2. How far away was that galaxy from us when it emitted the light we can see? - the answer is *S/Z*.
3. How far has light travelled in journeying from that galaxy to us? - the answer to this question is $c\Delta T$.

Here is a table which lists the three different answers to these questions in the three different universes. (All distances are in Gly or billions of light years. For the sake of argument, we shall take the expansion time for the exponential universe τ to be 5 billion years (This is the figure needed to be consistent with today's value of the Hubble Constant). We shall assume that the age of the linear universe is 13.6 billion years but we shall take the age of the duotritaic universe to be 9.2 billion years. (As we shall see in the section on the Hubble constant, the duotritaic universe is 2/3 of the age of the equivalent linear universe.)

Z = 4	**Exponential U**	**Linear U**	**duotritaic U**
How far away is it now?	15	18.9	14.9
How far away was it then?	3.8	4.7	3.7
How far has light travelled in getting here?	6.9	10.2	8.0

If we repeat the calculations for the CMB:

Z = 1000	**Exponential U**	**Linear U**	**duotritaic U**
How far away is it now?	5000	94	27.6
How far away was it then?	5	0.094	0.028
How far has light travelled in getting here?	34	13.6	9.2

It is perfectly evident that answer to the original question depends heavily on what kind of universe you think we are living in and what you meant by the question!

How far can we see?

It is as well to be reminded once again that all the universes we are considering so far are flat and therefore infinite. We can, however, (in principle) only see everything whose red shift is finite. So to answer the question, 'how far can we see?' we simply put $Z = \infty$ in the equations for S. As we has seen, this gives us ∞ for the exponential universe and the linear universe but only $3cT_0$ for the duotritaic one. This doesn't quite get to the very bottom of the question, though. Perhaps we should ask a slightly more subtle question. What fraction of the universe can we see and do we see more or less of it as time goes by?

In both the exponential universe and the linear universe, we can see an infinite volume of space – and therefore an infinite number of galaxies, but that does not mean that we can necessarily see everything in the universe. There are an infinite number of numbers between 0 and 1 but that is not all the numbers that exist. On the other hand, since none of the galaxies which may or may not be out there beyond $Z = \infty$ can ever influence us in any way (nor could they ever have influenced us in the past) they might as well not exist.

In the duotritaic universe, however, we can only see a volume of space out to $3cT_0$. and since the density of galaxies in space is finite, that means that at this instant in time we can only see a finite number of galaxies. Now consider the situation when the universe becomes twice as old. The universe will expand according to the duotritaic law by a factor equal to $2^{2/3} = 1.6$. The volume containing all the originally observable galaxies will therefore increase by a factor of $1.6^3 = 4$ – but the volume of the *observable* universe will increase by a factor of 8. This means that we will now be able to observe twice the number of galaxies as we did before! As time proceeds, more and more galaxies will come into view, emerging like ghosts out of the fog of the CMB!

Even now we have not yet quite got to the bottom of the question. In practice we cannot hope to see galaxies whose red shift is greater than that of the CMB and the most distant galaxy which has currently been imaged by the HST has an estimated red shift of about 10. The CMB has a red shift of about 1000. The most significant question to ask is, how far can the HST see as a proportion of the distance to the CMB? The answer to this question is the ratio:

$$\frac{S_{Z=10}}{S_{Z=1000}} \tag{32}$$

and the answers are: in the exponential universe: about 1%; in the linear universe: about a third; and in the duotritaic universe: nearly three quarters. (In the last section I shall argue that the best answer to this question is probably the middle one.)

The Hubble Constant

Originally the Hubble Constant H_0 was defined as v/S where v is the recession speed of a galaxy at a distance S from us. We have seen that we can no longer attach any meaning to the recession speed of a galaxy. The only thing we can actually measure it its cosmological red shift. For galaxies reasonably close to us where $v << c$ and $z << 1$, we know that $z = v/c$, hence:

$$H_0 = \frac{v}{S} = \frac{cz}{S} \tag{33}$$

In the **exponential universe**

$$S = c\tau z \tag{34}$$

so

$$\boxed{H_0 = \frac{cz}{S} = \frac{1}{\tau}} \tag{35}$$

and since τ is a constant, the Hubble constant is truly constant – every where and for all time. And, of course, in this universe there was no Big Bang (or rather, the Big Bang was infinitely long ago).

What about the **linear universe**?

Equation (9) tells us that

$$S = cT_0 \ln(1 + z) \tag{36}$$

Now for small z, $\ln(1 + z) = z$ so

$$S = cT_0 z \tag{37}$$

$$H_0 = \frac{1}{T_0} \tag{38}$$

which, of course, is only what one would intuitively expect for a universe which expands linearly with time.

Given the current best estimate of H_0 = 71 km s^{-1} / Mpc, this translates into an age for the universe of 4.3 × 10^{17} s or 13.6 billion years.

In the **duotritaic universe:**

For small z the equation reduces to
$$S = 3cT_0(z/2) \tag{39}$$

hence
$$H_0 = \frac{2}{3}\frac{1}{T_0} \tag{40}$$

Using the same value for the Hubble constant we see that the age of such a universe is a little smaller being only 9.2 billion years. (This is a bit of a problem for this model as some globular clusters are thought to be older than this!)

The Cosmic Microwave Background

It is well known that the Cosmic Microwave Background radiation (CMB) has a temperature of 2.7 K and it is believed that it results from the photons released (decoupled) when the universe became transparent. The temperature of the universe was then about 3000 K.

Now the peak wavelength λ_{max} emitted by a black body at a temperature τ is given by Wien's law

$$\lambda_{max} = \frac{b}{\tau} \tag{41}$$

So if the temperature has dropped from 3000 K to 2.7 K, the wavelength of the photons has been increased by the same amount. This means that the red shift of the CMB is equal to Z = 3000 / 2.7 = 1110

Now we can easily calculate when the decoupling happened. In a linearly expanding universe it is simply 13.6 Gyr / 1110 = 12 Myr. In a duotritaic universe, it is 9.2 Gyr / $(1110)^{3/2}$ = 290,000 years. Neither of these figures agree with the widely quoted figure of 370,000 years though the latter is a lot closer. To resolve this issue we must now consider the $64,000 question:

What kind of universe do we actually live in?

Having discussed in great detail the properties of three different types of universe, I am going to have to come clean. We don't live in any of them! In the first place, all the universes we have been considering are 'flat'. What this means is that the amount of matter in them is just sufficient to make the curvature zero. Under these circumstances the density of matter in the universe has a particular critical value. Going back to the original Friedmann equations, the first one tells us that:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{1}{3}\Lambda + \frac{8\pi G}{3}\rho - k\frac{c^2}{a^2} \tag{42}$$

So for a flat universe in which the cosmological constant is zero, both Λ and k are zero and the formula reduces to

$$H^2 = \frac{8\pi G}{3}\rho_{\text{crit}} \tag{43}$$

where ρ_{crit} is the critical density. Plugging in the currently accepted value of the Hubble constant we can easily calculate that the critical density for our universe is 9.7 × 10^{-27} kg m^{-3} which is about 6 hydrogen atoms per cubic metre. This may not sound an awful lot but remember, there is an awful lot of space out there. If we try to estimate the total amount of matter in the universe which we can actually see or infer (stars, dust, black holes and galactic dark matter etc.) we are hard pressed to find even 5% of this amount. So if our estimate of the amount of matter is correct, either the cosmological constant is not zero or *k* is negative. (The current fashion is, however, to assume that *k* is, in fact, zero; that Λ is positive but not big enough to make up the difference and that there is a large amount of extra dark matter out there to make the density of the universe exactly equal to the critical density.)

In the second place, according to the currently accepted model our universe is going through a number of different phases. First a phase in which it was dominated by radiation (and during which it may have undergone a period of very rapid expansion called inflation). Then there was a phase which was dominated by matter (a duotritaic phase). Finally we appear to be entering a phase in which the cosmological constant is becoming dominant (an exponential phase).

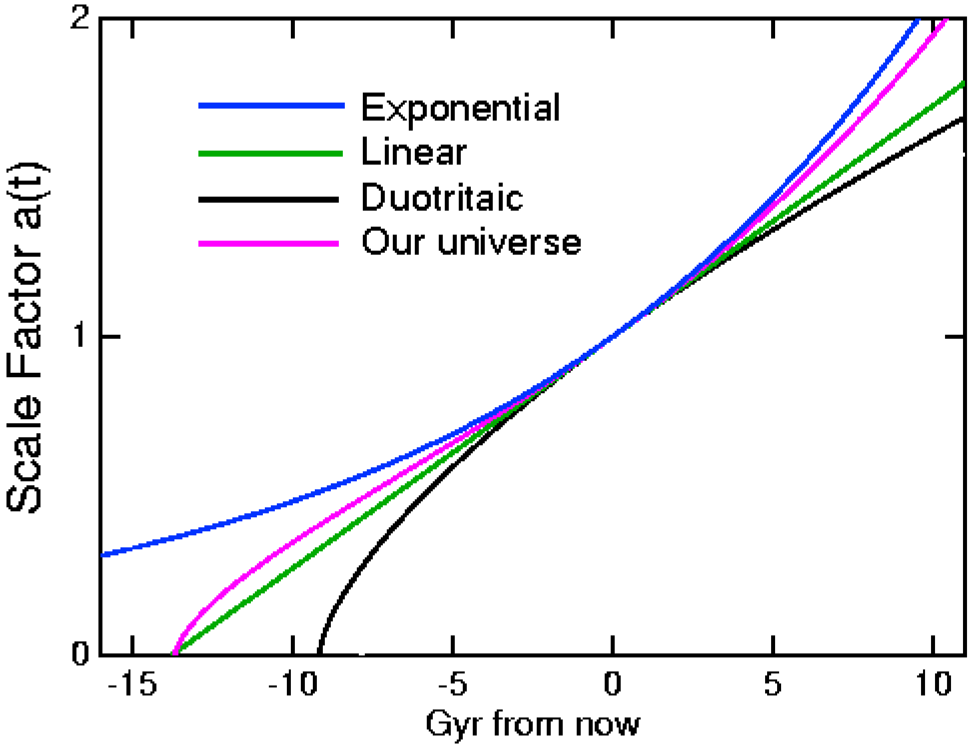

Figure 3: Showing how the history of our universe may have compared with those of the three models considered

By jiggling with various parameters, astronomers have come up with a model of a flat universe which follows the magenta line in figure 3 and which (by complete coincidence) happens to start at $T = 13.6$ billion years BP – almost exactly the same age as predicted by the linear model. It starts by expanding according to the matter-dominated model (which is why the age of the decoupling event is a lot closer to the figure predicted by the duotritaic universe) but then starts to curve back up again as the effect of the cosmological constant (or dark energy) begins to kick in. (All four curves have the same gradient at $T = now$ so as to agree with the observed value of the Hubble constant.)

Your students' questions answered

If you have, with perfect justification, skipped all the above mathematics and just want some answers, you can do worse than to assume that our universe is 'flat'; that it started in a Big Bang some 13.6 billion years ago and that, after a short period of rapid expansion at the start, it has been expanding approximately linearly ever since. The universe became transparent about 370,000 years after the Big Bang and the photons which set out on their journey towards us at that time have travelled 13.6 billion light years to reach us but, because of the expansion of the universe during that time, what we see as the Cosmic Background Radiation is in fact something like 90 billion light years away at the moment.

Because the universe is 'flat', the universe is infinitely large (and always has been). It is a bit misleading, therefore, to say that the universe is expanding. On the other hand, it is undoubtedly true that the universe is less dense now that it was . Moreover, the CMB acts as a sort of 'horizon' which limits what we call the 'observable' universe. As time goes by, the observable universe will increase in size and (theoretically) we should be able to observe new galaxies forming as they emerge into view.

The most distant object we have actually observed so far has a red shift of about 10. This puts it 30 billion light years away – or about one third of the distance to the edge of the observable universe.

If we are wrong about the universe being flat, there are two possibilities. Either the total mass of the universe is less than its critical mass or it is greater. In the former (more likely) case, the universe will fly apart faster and faster in the future. This implies that it expanded less quickly in the past than our calculations suggest and that the universe may be older than 13.6 billion years. This would please the astrophysicists who are already hard pressed to fit the evolution of the universe into the time scale allowed. Indeed, the situation may be compared to the the situation as regards the question of the age of the Earth before the discovery of radioactivity when calculations based on the cooling of the Earth lead to conclusions which were incompatible with the discoveries of geologists and evolutionary biologists. Almost nobody thinks that the mass of the universe is greater than its critical mass so it seems very unlikely that the universe is either finite or that it will end in a 'Big Crunch'.

I hope that these answers will satisfy most of your students – but perhaps the best answer to give to your most persistent and brightest students is the following: “Quite honestly, we don't know. Why don't you study physics or astronomy at university and help us find out?”

References

[1] For an authoritative and very readable account of the current state of thinking in cosmology, I cannot recommend the following article too highly: http://wmap.gsfc.nasa.gov/universe/WMAP_Universe.pdf

[2] Eugenio Bianchi, Carlo Rovelli :Why all these prejudices against a constant?

Centre de Physique Theorique de Luminy, Case 907, F-13288 Marseille, EU

(Dated: February 23, 2010) http://arxiv.org/abs/1002.3966

For further insights into how General Relativity is applied to the cosmos I strongly recommend:

[3] 'Ned Wright's Cosmology Tutorial' http://www.astro.ucla.edu/~wright/cosmo_01.htm

For a fascinating interactive applet showing how different models of the universe evolve with time see:

[4] J.P.Leahy 'The simplest models of the universe' http://www.jb.man.ac.uk/~jpl/cosmo/friedman.html

Figures

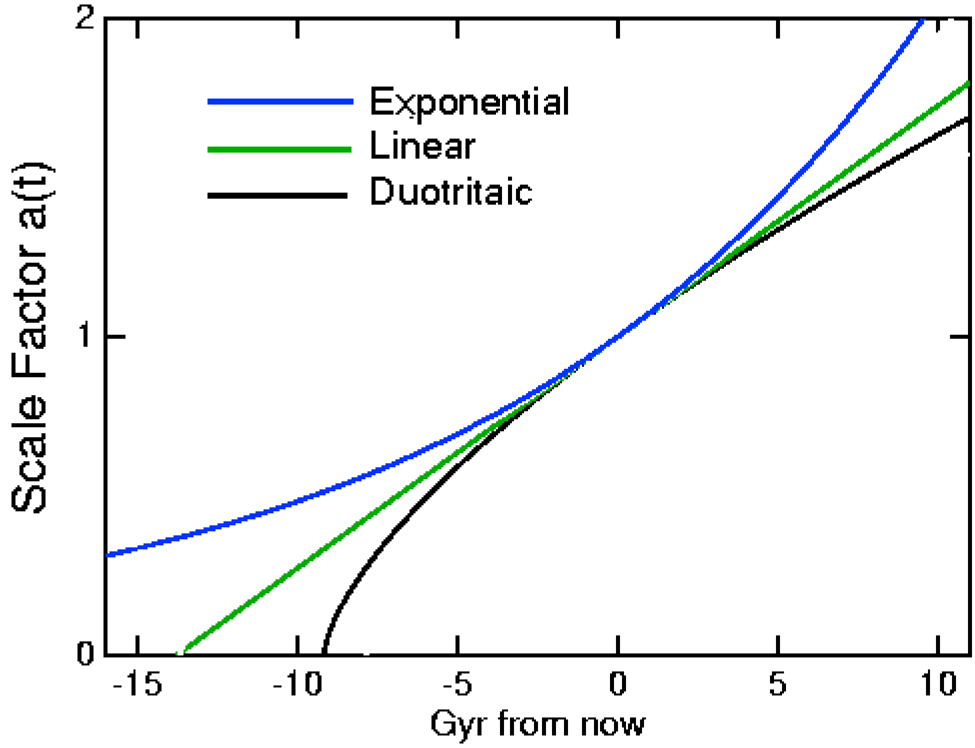

figure (1): The expansion of the universe according to three different models. All three models are 'flat' (ie $k = 0$). The exponential universe is dominated by the cosmological constant; the duotritaic universe is dominated by matter.

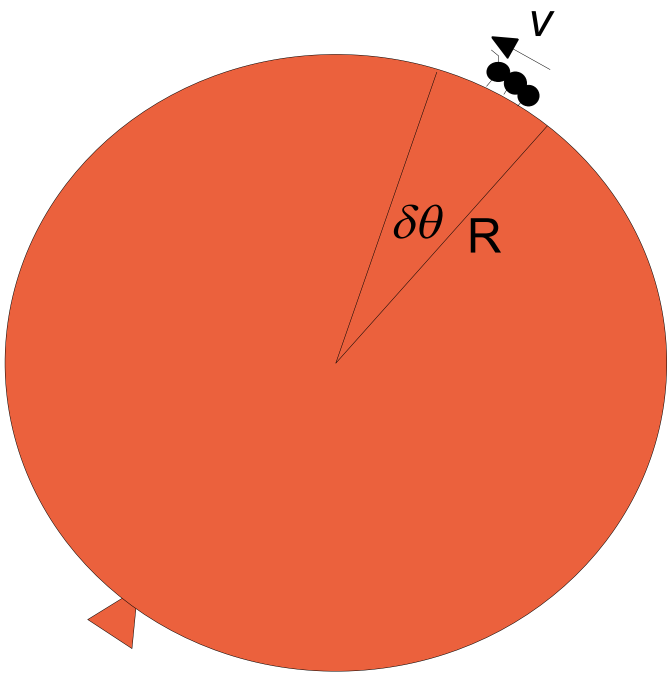

figure (2): An ant walking round an expanding balloon.

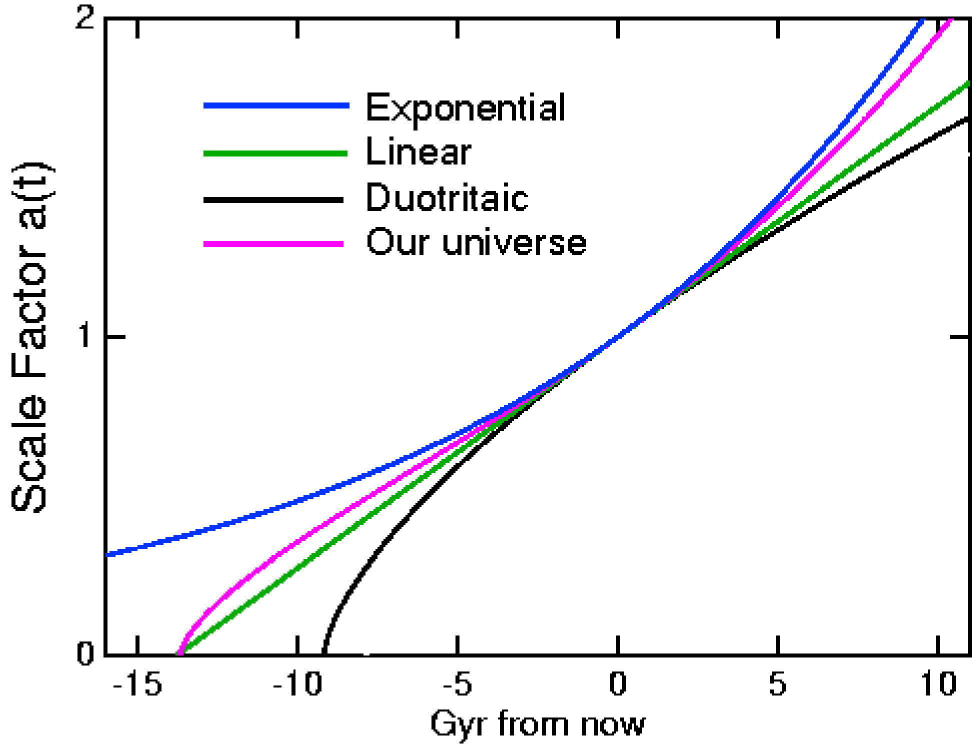

figure (3): Showing how the history of our universe may have compared with those of the three models considered